Piecewise Deterministic Markov Processes and (Ecological) Applications

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Talk based on results obtained over the recent years in collaboration with several colleagues

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– Bakhtin & Hurth, (Nonlinearity, 2012)
– Benaïm, Le Borgne, Malrieu, & Zitt (ECP 2012; AAP 2014; IHP 2015)
– B & Lobry (AAP 2016)
– B, Colonius, & Lettau (Nonlinearity 2017)
– B, H, & Strickler (ECP 2018)
– Bakhtin, H, & Mattingly (Nonlinearity 2015)
– Bakhtin, H, Lawley, & Mattingly (Nonlinearity 2018)
– B & Strickler (AAP 2019)
– B & H (2021)

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Here we restrict attention to the following specific class:

- $E = \{1, \ldots, m\}$,
- $F^1, \ldots, F^m$ smooth vector fields on $\mathbb{R}^d$,
- $(\Phi^1_t), \ldots, (\Phi^m_t)$ induced flows
- $M \subset \mathbb{R}^d$ positive invariant set under each Φ^i,
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- $M \subset \mathbb{R}^d$ positive invariant set under each Φ^i,

- For $x \in M$, $(Q_{ij}(x))$ Markov transition matrix over E (irreducible, aperiodic, and continuous in x) : switching mechanism.

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– Repeat.

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However, (Z_t) is not strong Feller!

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Discuss some applications to ecological and/or epidemic models

Outline

Motivating Examples

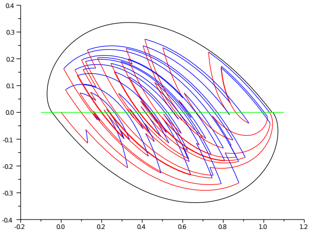
A linear example

$E = \{1,2\}, M = \mathbb{R}^2$

$$F^1(x) = Ax, F^2(x) = A(x-e)$$

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$\lambda = 1, Q_{ij}(x) = 1/2.$$



Lotka-Volterra (based on B & Lobry, AAP 2016)

$E = \{1, 2\}, M := \mathbb{R}_+ \times \mathbb{R}_+.$

F^1, F^2 two competitive Lotka-Volterra vector fields

$$F^i(x, y) = \begin{cases} \alpha_i x(1 - a_i x - b_i y) \\ \beta_i y(1 - c_i x - d_i y) \end{cases}$$

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both favorable to the same species x:

$$a_i < c_i \text{ and } b_i < d_i.$$

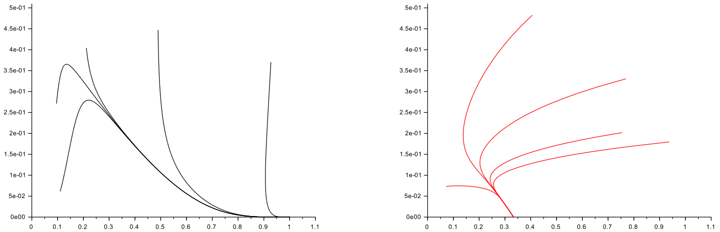

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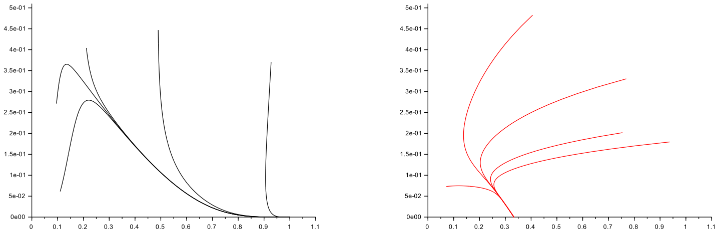
Figure: Phase portraits of F^1 and F^2



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$$Q(x) = \begin{pmatrix} 1-p & p \\ 1-p & p \end{pmatrix}.$$

Different values of p, λ lead to different behaviors $\hookrightarrow$

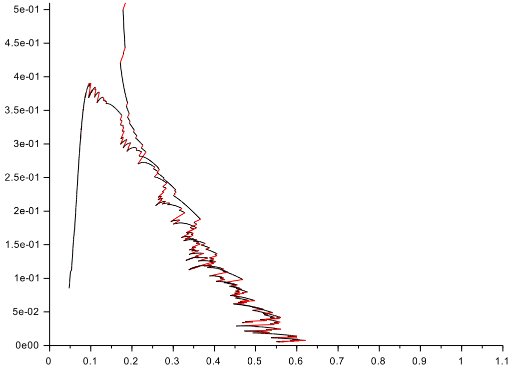


Figure: Extinction of y

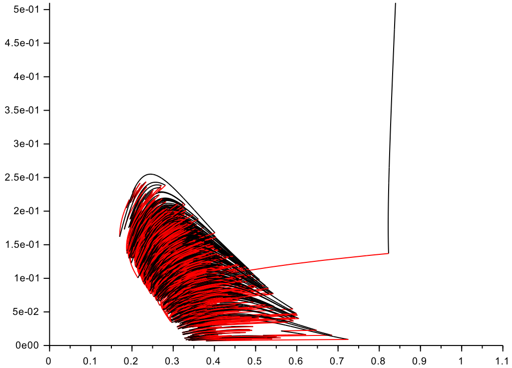
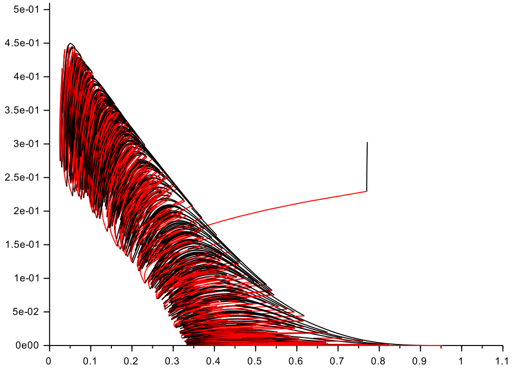


Figure: Persistence



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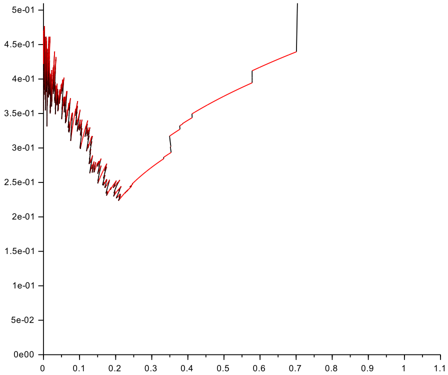

Figure: Extinction of x

How is this possible?

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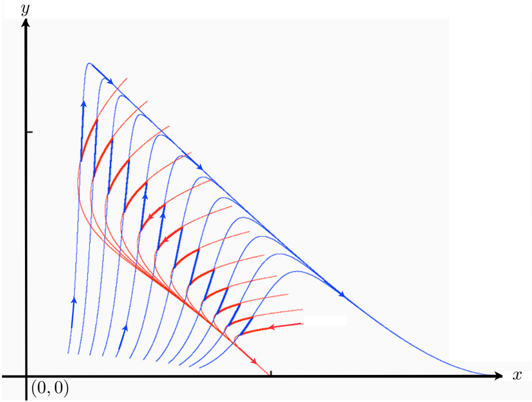
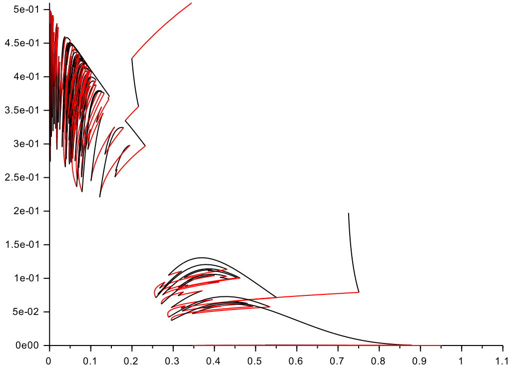
Have look at this picture:





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Some Math

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The results here are mainly based on
– Bakhtin & H (Nonlinearity 2012)
– B, Le Borgne, Malrieu, Zitt, (Annales IHP 2015)
– B, H, & Strickler (ECP 2018)
– B & H (2021)

A support theorem

$$co(F)(x) := \mathrm{conv}(F^1(x), \ldots, F^m(x)),$$

• S^x is the set of (absolutely continuous) maps $\eta : \mathbb{R}_+ \to \mathbb{R}^d$ solutions to

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- equivalently:

$$\eta \in S^x \Leftrightarrow \dot{\eta}(t) = \sum_{i=1}^{m} u^i(t) F^i(\eta(t)), \eta(0) = x$$

with $u^i \in L^\infty, u^i \geq 0, \sum u^i(t) = 1.$

Clearly

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Theorem

If $X_0 = x$, then the support of the law of (X_t) equals S^x.

Invariant probability measures

A probability measure on $M \times E$ is called *invariant* for (Z_t) whenever

$$Law(Z_0) = \mu \Rightarrow Law(Z_t) = \mu$$

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Ergodic measure = extremal invariant probability measure

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• except in dimension 1: (Bakhtin, H, and Mattingly, Nonlinearity 2015)

• and for particular 2-dimensional systems (Bakhtin, H, Lawley, and Mattingly, Nonlinearity 2018)

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Proposition

(i) $\Gamma \times E \subset supp(\mu)$ *for all* μ *invariant.*

(ii) *If* Γ *has nonempty interior, then* $\Gamma \times E = supp(\mu)$ *for all* μ *invariant.*

(iii) *If* Γ *is compact, there exists* μ *invariant such that* $\Gamma \times E = supp(\mu)$.

(iv) *If* $\exists!$ *invariant probability measure* μ, *then* $\Gamma \times E = supp(\mu)$.

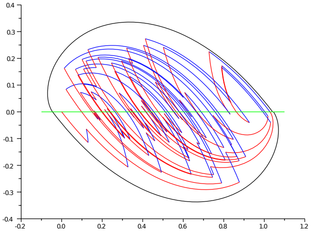

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Figure: Example of accessible set

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Furstenberg (1961) showed that there exists a smooth diffeo on the torus, preserving the area, topologically transitive but not uniquely ergodic.
$\Rightarrow$ One can construct a flow with the same properties.

$\Rightarrow$ Natural question: Under which conditions is there (at most) one invariant probability measure?

Uniqueness and weak bracket

The Weak Bracket condition

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Weak Bracket at $x \in M$:

For some $k \geq 1, \mathbf{F}_k$ has full rank at x.

Uniqueness

Theorem

Suppose $\exists x \in \Gamma$ *at which the* **weak bracket** *condition holds. Then there is at most one invariant probability measure* μ.

If μ *exists, it is absolutely continuous with respect to Lebesgue measure and* $\forall f \in L^1(\mu), z \in M \times E$,

$$\mathbb{P}_z\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t f(Z_s)ds = \mu(f)\right) = 1.$$

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The existence is not guaranteed in general, but is ok when M is compact or under the existence of a suitable Lyapunov function

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Indeed,

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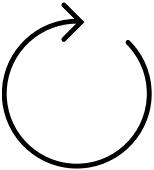

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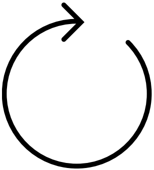

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Here $\mu =$ Lebesgue, but $Law(Z_t) = \delta_{(\Phi_t(x),1)} \not\rightarrow \mu$

Convergence and strong bracket

The Strong Bracket condition

$$\mathbf{G}_0 = \{F^i - F^j \ : i, j = 1, \ldots m\},$$

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Theorem

Suppose $\exists p \in \Gamma$ at which the **strong bracket** *condition holds. Then $\exists C, \kappa > 0$ such that for all f measurable*

$$|P_t f(z) - \mu(f)| \leq Ce^{-\kappa t}(1 + V(z))\|f\|_V.$$

Here $\|f\|_V = \sup \frac{|f(z)|}{1+V(z)}$.

Convergence

An alternative condition (sometimes very useful):

Theorem

Suppose $\exists p \in \Gamma$ *at which the* **weak bracket** *condition holds and* $\exists q \in \Gamma$ *at which a barycentric combination of the* F^i *vanishes. Then* $\exists C, \kappa > 0$ *such that for all* f *measurable*

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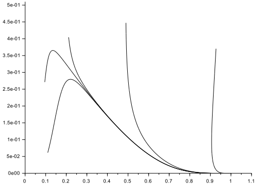
Application to LV

$E = \{1, 2\}, M := \mathbb{R}_+ \times \mathbb{R}_+.$
F^1, F^2 two competitive LV favorable to x

$$F^i(x, y) = \begin{cases} \alpha_i x(1 - a_i x - b_i y) = xU^i(x, y) \\ \beta_i y(1 - c_i x - d_i y) = yV^i(x, y) \end{cases}$$



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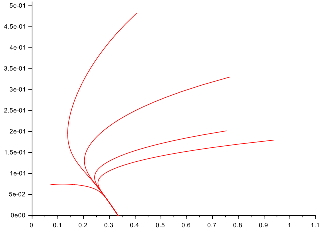

Invasion rates

- On the "face" $\{y = 0\}$ the system is a $1D$ PDMP obtained by switching between the ode's

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Invasion rate of y:

$$\Lambda_y = \int V^i(x,0)\nu(dxdi).$$

A persistence theorem

- We suppose that $\Lambda_x, \Lambda_y > 0$.
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Persistence Theory (not the subject today) $\Rightarrow$

For the system restricted to $M_0 \times E$ (not $M \times E$!) the map

$$V(x, y, i) = \frac{1}{x^\theta} + \frac{1}{y^\theta}$$

is a Lyapunov function for some $\theta > 0$.

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Theorem (B, Lobry 16 + B, H, Strickler 19)

There exists a unique invariant probability measure Π *on* $M_0 \times E$*, absolutely continuous, and* $\forall z \in M_0 \times E$

$$\|\mathbb{P}_z(Z_t \in \cdot) - \Pi(\cdot)\| \leq C(1 + x^{-\theta} + y^{-\theta})e^{-\kappa t}$$

with $C, \kappa > 0$.
Moreover,

$$supp(\Pi) = \Gamma \times E$$

and Γ *is simply connected.*

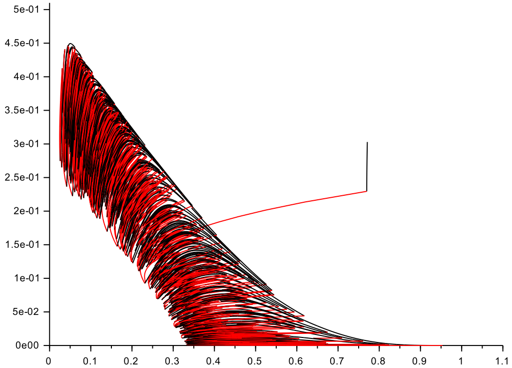

Figure: The set Γ