Approximation of Invariant Measures for Regime-Switching Diffusions

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- Yuan, C. and Mao, X., Asymptotic stability in distribution of stochastic differential equations with Markovian switching. Stochastic Process. Appl. 103 (2003), 277-291.
- Bao, J., Shao, J., Yuan, C., Approximation of invariant measures for regime-switching diffusions. Potential Anal. 44 (2016), 707-727.
- Li, X., Ma, Q., Yang, H., Yuan, C., The numerical invariant measure of stochastic differential equations with Markovian switching. SIAM J. Numer. Anal. 56 (2018), 1435-1455.

Outline

- Motivations
- Invariant Measure of regime-switching diffusion processes
- Numerical Invariant Measure with Lipschitz condition
- Numerical Invariant Measure without Lipschitz condition

Motivations

- A regime-switching diffusion process (RSDP), is a diffusion process in random environments characterized by a Markov chain.
- The state vector of a RSDP is a pair $(X(t), \Lambda(t))$, where $\{X(t)\}_{t\geq 0}$ satisfies a stochastic differential equation (SDE)

$$\mathrm{d}X(t) = b(X(t), \Lambda(t))\mathrm{d}t + \sigma(X(t), \Lambda(t))\mathrm{d}W_t, \quad t > 0, \tag{1}$$

with the initial data $X_0 = x \in \mathbb{R}^n, \Lambda_0 = i \in \mathbb{S}$, and $\{\Lambda(t)\}_{t\geq 0}$ denotes a continuous-time Markov chain with the state space $\mathbb{S} := \{1, 2 \cdots, N\}$, $1 \leq N \leq \infty$, and the transition rules specified by

$$\mathbb{P}(\Lambda(t+\triangle) = j | \Lambda(t) = i) = \begin{cases} q_{ij}\triangle + o(\triangle), & i \neq j, \\ 1 + q_{ii}\triangle + o(\triangle), & i = j. \end{cases} \tag{2}$$

Motivations (Cont.)

- RSDPs have considerable applications in e.g. control problems, storage modeling, neutral activity, biology and mathematical finance The dynamical behavior of RSDPs may be markedly different from diffusion processes without regime switchings, see e.g.

Mao, X. and Yuan, C., *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, 2006.

Pinsky, M., Scheutzow, M., Some remarks and examples concerning the transience and recurrence of random diffusions, *Ann. Inst. Henri. Poincaré*, **28** (1992), 519–536.

Yin, G., Zhu, C., *Hybrid Switching Diffusions: Properties and Applications*, Springer, 2010.

It is interesting to have a look of the following two equations

$$dx(t) = x(t)dt + 2x(t)dW(t) \tag{3}$$

and

$$dx(t) = 2x(t) + x(t)dW(t) \tag{4}$$

switching from one to the other according to the movement of the Markov chain $\Lambda(t)$. We observe that Eq. (3) is almost surely exponentially stable since the Lyapunov exponent is $\lambda_1 = -1$ while Eq. (4) is almost surely exponentially unstable since the Lyapunov exponent is $\lambda_2 = 1.5$.

Let $\Lambda(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with the generator

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -1 & 1 \\ \gamma & -\gamma \end{pmatrix}.$$

Of course $W(t)$ and $\Lambda(t)$ are assumed to be independent. Consider a one-dimensional linear SDEwMS

$$dx(t) = a(\Lambda(t))x(t)dt + b(\Lambda(t))x(t)dW(t) \tag{5}$$

on $t \geq 0$, where

$$a(1) = 1, \quad a(2) = 2, \quad b(1) = 2, \quad b(2) = 1.$$

However, as the result of Markovian switching, the overall behaviour, i.e. Eq. (5) will be exponentially stable if $\gamma > 1.5$ but exponentially unstable if $\gamma < 1.5$ while the Lyapunov exponent of the solution is 0 when $\gamma = 1.5$.

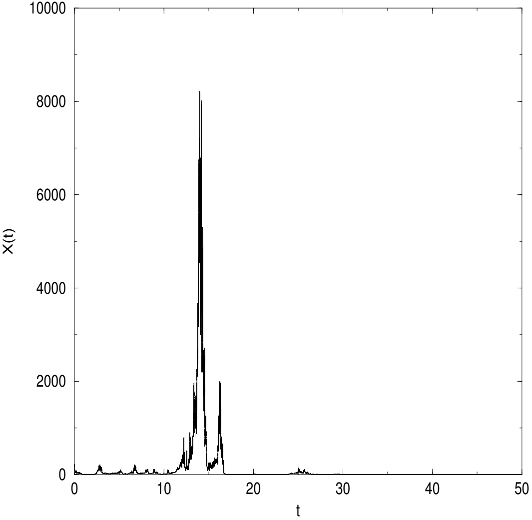

Figure: The graph of numerical solution when $\gamma = 2$.

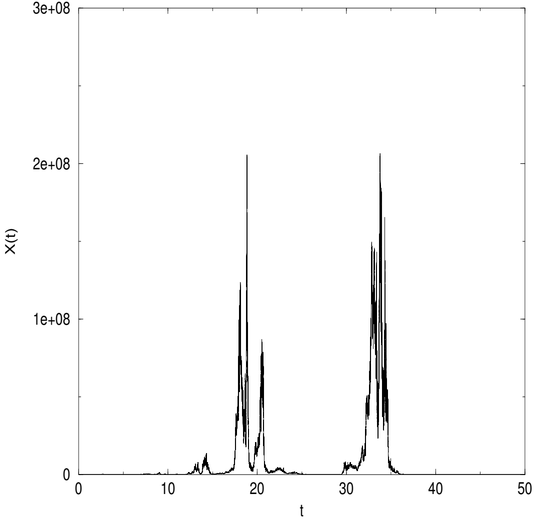

Figure: The graph of numerical solution when $\gamma = 1.5$.

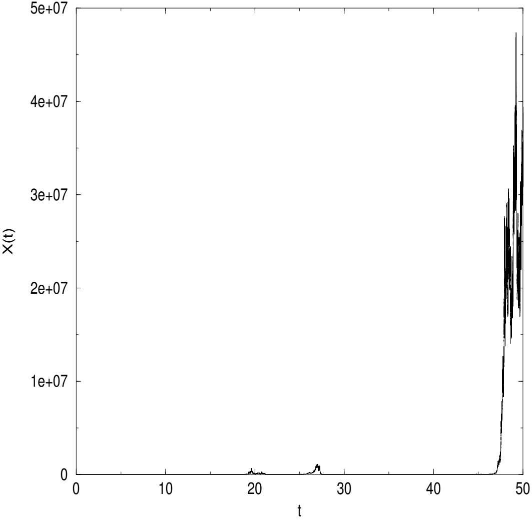

Figure: The graph of numerical solution when $\gamma = 0.5$.

Motivations (Cont.)

- Since solving RSDPs is still a challenging task, numerical schemes and/or approximation techniques have become one of the viable alternatives (see e.g. Mao-Yuan (2006), Yin-Zhu (2010), Higham et al. (2007)).
- For more details on numerical analysis of diffusion processes without regime switching, please refer to the monograph by Kloeden and Platen (1992).
- Also, approximations of invariant measures for stochastic dynamical sysytems have attracted much attention, see e.g. Mattingly et al. (2010), Talay (1990), Bréhier (2014).

Motivations (Cont.)

- In this talk, we are concerned with the following questions:

(i) Under what conditions, will the semigroup of the exact solution admit an invariant measure?

(ii) Under what conditions, will the discrete-time semigroup generated by EM scheme admit an invariant measure?

(iii) Will the numerical invariant measure, if it exists, converge in some metric to the underlying one?

Invariant Measure

Some notation is listed as below.

- $(\Omega, \mathscr{F}, \mathbb{P})$: probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$;
- $\{W_t\}_{t\geq 0}$: an m-dimensional Brownian motion;
- $\{\Lambda(t)\}_{t\geq 0}$: a continuous-time Markov chain with the state space $S := \{1, 2 \cdots, N\}, N < \infty$; The transition rules of $\{\Lambda(t)\}_{t\geq 0}$ are specified by (2)
- Q-matrix $Q := (q_{ij})_{N\times N}$ is irreducible and conservative so that $\{\Lambda(t)\}_{t\geq 0}$ has a unique stationary distribution $\mu := (\mu_1, \cdots, \mu_N)$.

We assume that, in (1), $b : \mathbb{R}^n \times \mathbb{S} \mapsto \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times \mathbb{S} \mapsto \mathbb{R}^n \otimes \mathbb{R}^m$ satisfy the following condition

(H0) *Both b and σ satisfy the local Lipschitz condition and the linear growth condition. That is, for each $k = 1, 2, \ldots$, there is an $h_k > 0$ such that*

$$|b(x,i) - b(y,i)| + |\sigma(x,i) - \sigma(y,i)| \leq h_k|x - y|$$

for all $i \in S$ and those $x, y \in \mathbb{R}^n$ with $|x| \vee |y| \leq k$; and there is moreover an $h > 0$ such that

$$|b(x,i)| + |\sigma(x,i)| \leq h(1 + |x|)$$

for all $x \in \mathbb{R}^n$ and $i \in S$.

If $V \in C^2(\mathbb{R}^n \times S; \mathbb{R}_+)$, define an operator LV from $\mathbb{R}^n \times S$ to $\mathbb{R}$ by

$$LV(x,i) = \sum_{j=1}^{N} \gamma_{ij} V(x,j) + V_x(x,i) b(x,i) + \frac{1}{2} \mathrm{trace}[\sigma^T(x,i) V_{xx}(x,i) \sigma(x,i)] \tag{6}$$

where

$$V_x(x,i) = \left(\frac{\partial V(x,i)}{\partial x_1}, \ldots, \frac{\partial V(x,i)}{\partial x_n} \right), \quad V_{xx}(x,i) = \left(\frac{\partial^2 V(x,i)}{\partial x_i \partial x_j} \right)_{n \times n}.$$

$$
\begin{aligned}
X^{x,i}(t) - X^{y,i}(t) &= x - y + \int_0^t [b(X^{x,i}(s), \Lambda_i(s)) - b(X^{y,i}, \Lambda_i(s))]ds \\
&+ \int_0^t [\sigma(X^{x,i}(s), \Lambda_i(s)) - \sigma(X^{y,i}, \Lambda_i(s))]dB(s).
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}U(x,y,i) &= \sum_{j=1}^{N} \gamma_{ij} U(x-y,j) + U_x(x-y,i)[b(x,i) - b(y,i)] \\
&+ \frac{1}{2}\mathrm{trace}\Big([\sigma(x,i) - \sigma(y,i)]^T U_{xx}(x-y,i)[\sigma(x,i) - \sigma(y,i)]\Big).
\end{aligned}
$$

Let $y(t)$ denote the $\mathbb{R}^n \times S$-valued process $(X(t), \Lambda(t))$. Then $y(t)$ is a time homogeneous Markov process. Let $p(t, x, i, dy \times \{j\})$ denote the transition probability of the process $y(t)$.

Let $\mathcal{P}(\mathbb{R}^n \times S)$ denote all probability measures on $\mathbb{R}^n \times S$. For $P_1, P_2 \in \mathcal{P}(\mathbb{R}^n \times S)$ define metric d_{Lip} as follows:

$$d_{\mathrm{Lip}}(P_1, P_2) = \sup_{f \in \mathrm{Lip}} \Big| \sum_{i=1}^{N} \int_{\mathbb{R}^n} f(x,i) P_1(dx, i) - \sum_{i=1}^{N} \int_{\mathbb{R}^n} f(x,i) P_2(dx, i) \Big| \tag{7}$$

and

$$\mathrm{Lip} = \{ f : \mathbb{R}^n \times S \to \mathbb{R} : |f(x,i) - f(y,j)| \le |x-y| + |i-j| \text{ and } |f(\cdot,\cdot)| \le 1 \}. \tag{8}$$

Theorem (Theorem 1, Mao, Y., (2003))

Assume that there exist functions $V, U \in C^2(\mathbb{R}^n \times S; \mathbb{R}_+)$, $\mu, \mu_1, \mu_2 \in \mathcal{K}_\infty$ *and positive numbers* β *and* λ_1 *such that*

$$\mu(|x|) \le V(x,i),\ U(0,i) = 0,\ \mu_1(|x|) \le U(x,i), \tag{9}$$

$$LV(x,i) \le -\lambda_1 V(x,i) + \beta,\ \mathcal{L}U(x,y,i) \le -\mu_2(|x-y|), \tag{10}$$

for all $(x,y,i) \in \mathbb{R}^n \times \mathbb{R}^n \times S$. *Then there exists a probability measure* $\pi(\cdot \times \cdot)$ *on* $\mathbb{R}^n \times S$ *such that the transition probability* $p(t,x,i,dy \times \{j\})$ *of* $y(t)$ *converges weakly to* $\pi(dy \times \{j\})$ *as* $t \to \infty$ *for every* $(x,i) \in \mathbb{R}^n \times S$. *(*$y(t)$ *is called asymptotically stable in distribution.)*

Theorem (Theorem 2, Mao, Y., (2003))

Let (H0) hold. Assume that for all $x, y \in \mathbb{R}^n$ *and* $i \in S$,

$$x^T b(x,i) \le \beta_i |x|^2 + \alpha, \tag{11}$$

$$(x-y)^T (b(x,i) - b(y,i)) \le \beta_i |x-y|^2, \tag{12}$$

$$|\sigma(x,i)|^2 \le \delta_i |x|^2 + \alpha, \tag{13}$$

$$|\sigma(x,i) - \sigma(y,i)|^2 \le \delta_i |x-y|^2. \tag{14}$$

where α, β_i *and* δ_i *are constants. If*

$$\mathcal{A} := -\mathrm{diag}(2\delta_1 + \delta_1, \cdots, 2\beta_N + \delta_N) - Q \tag{15}$$

is an M-matrix, then $y(t)$ *is asymptotically stable in distribution.*

We assume that

(H)

$$|b(x,i)-b(y,i)|+\|\sigma(x,i)-\sigma(y,i)\| \leq L_R|x-y|, \quad x,y \in B_R(0). \tag{16}$$

For each $i \in \mathbb{S}$ and $x,y \in \mathbb{R}^n$, there exist $c_0 > 0$ and $\beta_i \in \mathbb{R}$ such that

$$2\langle x, b(x,i)\rangle + \|\sigma(x,i)\|^2 \leq c_0 + \beta_i|x|^2, \tag{17}$$

$$2\langle x-y, b(x,i)-b(y,i)\rangle + \|\sigma(x,i)-\sigma(y,i)\|^2 \leq \beta_i|x-y|^2. \tag{18}$$

For any $p \in (0,1]$, define a metric $d_p(\cdot,\cdot)$ on $\mathbb{R}^n \times \mathbb{S}$ as below

$$d_p((x,i),(y,j)) = |x-y|^p + \mathbf{1}_{\{i \neq j\}}, \quad (x,i),(y,j) \in \mathbb{R}^n \times \mathbb{S},$$

where $\mathbf{1}_A$ denotes the indicator function of the set A. For $p \in (0,1]$, define the Wasserstein distance between $\nu \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ and $\nu' \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ by

$$W_p(\nu,\nu') = \inf \mathbb{E} d_p(X_1, X_2), \tag{19}$$

where the infimum is taken over all pairs of random variables X_1, X_2 on $\mathbb{R}^n \times \mathbb{S}$ with respective laws ν, ν'.

Invariant Measure (Cont.)

Theorem (Theorem 3, Bao, Shao, Y. (2016))

*Let $N < \infty$ and assume that (**H**) and*

$$\sum_{i=1}^{N} \mu_i \beta_i < 0. \tag{20}$$

Then $(X(t)^{x,i}, \Lambda(t)^i)$ admits a unique invariant measure $\pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$.

$\pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ is called an invariant measure of $(X(t)^{x,i}, \Lambda(t)^i)$ if

$$\pi(\Gamma \times \{i\}) = \sum_{j=1}^{N} \int_{\mathbb{R}^n} P_t(x, j; \Gamma \times \{i\}) \pi(\mathrm{d}x \times \{j\}), \quad t \geq 0.$$

Example I Let $\{\Lambda(t)\}_{t\geq 0}$ be a right-continuous Markov chain taking values in $\mathbb{S} := \{0,1\}$ with the generator

$$Q = \begin{pmatrix} -4 & 4 \\ \gamma & -\gamma \end{pmatrix}$$

with some $\gamma > 0$. Consider the scalar Ornstein-Uhlenback (O-U) process with regime switching

$$\mathrm{d}X(t) = \alpha_{\Lambda(t)}X(t)\mathrm{d}t + \sigma_{\Lambda(t)}\mathrm{d}W_t, \quad t > 0, \quad X_0 = x, \ \Lambda_0 = i_0 \in \mathbb{S}, \quad (21)$$

Example I (Cont.)

where $\alpha_\cdot : \mathbb{S} \mapsto \mathbb{R}$ such that $\alpha_0 = 1$, and $\alpha_1 = -1/2$, and $\sigma \in \mathbb{R}$ is a constant. The stationary distribution of $\{\Lambda(t)\}_{t\geq 0}$ admits the following form

$$\mu = (\mu_0, \mu_1) = \Big(\frac{\gamma}{4+\gamma}, \frac{4}{4+\gamma}\Big).$$

Let $\beta_0 = 2$, $\beta_1 = -1$, then **(H)** and (20) holds with $\gamma \in (0, 2)$. Hence $(X(t)^{x,i}, \Lambda(t)^i)$ admits a unique invariant measure $\pi \in \mathcal{P}(\mathbb{R} \times \mathbb{S})$ whenever $\gamma \in (0, 2)$.

Remark:

- By an M-Matrix approach, (21) has a unique invariant measure for $\gamma \in (0, 1)$, see e.g. Example 5.1 (Yuan-Mao, 2003).

Numerical Invariant Measure

For a given stepsize $\delta \in (0,1)$, define the discrete-time EM scheme associated with (1) as follows

$$\overline{Y}^{x,i}((k+1)\delta) := \overline{Y}^{x,i}(k\delta) + b(\overline{Y}^{x,i}(k\delta), \Lambda^i_{k\delta})\delta + \sigma(\overline{Y}^{x,i}(k\delta), \Lambda^i_{k\delta})\triangle W_k, \ k \geq 0, \quad (22)$$

with $\overline{Y}^{x,i}_0 = x, \Lambda^i_0 = i \in \mathbb{S}$, where $\triangle W_k := W((k+1)\delta) - W(k\delta)$ stands for the Brownian motion increment. Remark:

- $(\overline{Y}^{x,i}(k\delta), \Lambda^i_{k\delta})$ is a time homogeneous Markov chain.
- If $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ satisfies

$$\pi^\delta(\Gamma \times \{i\}) = \sum_{j=1}^N \int_{\mathbb{R}^n} P^\delta_{k\delta}(x, j; \Gamma \times \{i\}) \pi^\delta(\mathrm{d}x \times \{j\}), \ \Gamma \in \mathscr{B}(\mathbb{R}^n),$$

  then we call $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ an invariant measure of $(\overline{Y}^{x,i}(k\delta), \Lambda^i_{k\delta})$.

Numerical Invariant Measure: Additive Noise

we further assume that, for each $i \in \mathbb{S}$ and $x, y \in \mathbb{R}^n$, there exists an $L > 0$ such that

$$|b(x,i) - b(y,i)| + \|\sigma(x,i) - \sigma(y,i)\| \leq L|x-y|. \tag{23}$$

Theorem (Theorem 4, Bao, Shao, Y., (2016))

Let $N < \infty$, and assume further that (**H**), (20), *and* (23) *hold with* $\sigma(\cdot,\cdot) \equiv \sigma(\cdot)$. *Then,* $(\overline{Y}^{x,i}(k\delta), \Lambda^i_{k\delta})$ *admits a unique invariant measure* $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ *whenever the stepsize is sufficiently small.*

Theorem (Theorem 5 (Bao, Shao, Y., (2016))

Under conditions of **Theorem 4**, *for sufficiently small* $\delta \in (0,1)$,

$$W_p(\mu, \mu^\delta) \le c\delta^{p/2}, \quad p \in (0, 1 \wedge p_0),$$

where

$$W_p(\mu,\nu) := \inf_{\pi \in \mathcal{C}(\mu,\nu)} \int_{\mathbb{R}^n \times \mathbb{S}} \int_{\mathbb{R}^n \times \mathbb{S}} d(x,y)^p \pi(\mathrm{d}x, \mathrm{d}y), \quad p \in (0,1].$$

Numerical Invariant Measure: Multiplicative Noise

Theorem (Theorem 6, Bao, Shao, Y., (2016))

Let $N < \infty$, (23), *(**H**), and* (20) *hold. Assume further that*

$$\min_{i\in\mathbb{S}}\{-q_{ii}/\beta_i, \beta_i > 0\} > 1. \tag{24}$$

Then $(\overline{Y}^{x,i}(k\delta), \Lambda^i_{k\delta})$ *has a unique invariant measure* $\pi^\delta \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ *whenever the stepsize* $\delta \in (0,1)$ *is sufficiently small.*

Example II

Let $\{\Lambda(t)\}_{t\geq 0}$ be a right-continuous Markov chain taking values in $\mathbb{S} := \{0, 1, 2\}$ with the generator

$$Q = \begin{pmatrix} -(3+\nu) & \nu & 3 \\ 1 & -3 & 2 \\ 1 & 2 & -3 \end{pmatrix}$$

for some $\nu \geq 0$. Consider a scalar linear SDE with regime switching

$$\mathrm{d}X(t) = \alpha_{\Lambda(t)} X(t)\mathrm{d}t + \sigma_{\Lambda(t)} X(t)\mathrm{d}W_t, \quad t \geq 0, \quad X_0 = x, \ \Lambda_0 = i_0, \tag{25}$$

where $\alpha_\cdot, \sigma_\cdot : \mathbb{S} \mapsto \mathbb{R}$ such that

$$\alpha_0 = \frac{1}{2}, \alpha_1 = -2, \alpha_2 = -3, \qquad \sigma_0 = \frac{1}{3}, \sigma_1 = 2, \sigma_2 = 1.$$

Example II (Cont.)

Observe that (23) holds with $L = 4$, and (**H**) holds for $\beta_0 = \frac{10}{9}, \beta_1 = 0$, and $\beta_2 = -5$. Since the Markov chain possesses the stationary distribution

$$\mu = (\mu_0, \mu_1, \mu_2) = \Big(\frac{5}{20 + 5\nu}, \frac{6 + 3\nu}{20 + 5\nu}, \frac{9 + 2\nu}{20 + 5\nu}\Big),$$

it is easy to see that (20) and (24) are satisfied respectively for any $\nu \geq 0$. Then, $(\overline{Y}^{x,i}(k\delta), \Lambda^i_{k\delta})$ has a unique invariant measure for sufficiently small $\delta \in (0, 1)$.

Why Lipschitz condition for numerical solutions

D. Higham, X. Mao and C.Yuan, Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations. SIAM J. Numer. Anal. 45 (2007), 592–609.

For the scalar cubic SDE

$$dx(t) = \left(x(t) - x(t)^3\right) dt + 2x(t)dB(t), \tag{26}$$

we have

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t)| \le -1 \quad \text{a.s.} \tag{27}$$

The EM method

$$X_{k+1} = X_k \left(1 + \Delta t - \Delta t X_k^2 + 2\Delta B_k\right). \tag{28}$$

Lemma

Suppose $0 < \Delta t < 1$. *If* $|X_1| \ge 2^4/\sqrt{\Delta t}$, *then*

$$\mathbb{P}\left(|X_k| \ge \frac{2^{k+3}}{\sqrt{\Delta t}}\ \forall\, k \ge 1\right) \ge \exp\left(-4e^{-2/\sqrt{\Delta t}}\right).$$

Cont. without Lipschitz condition

(H1) For any $i \in \mathbb{S}$, there exists a constant $\alpha_i \in \mathbb{R}$ such that

$$(u-v)^T(b(u,i)-b(v,i)) \leq \alpha_i |u-v|^2, \quad \forall u,\ v \in \mathbb{R}^n.$$

Moreover, for any $R \geq 0$, there exists a positive constant L_R such that

$$|b(u,j)-b(v,j)| \leq L_R|u-v|,$$

for any $u,\ v \in \mathbb{R}^n$, $|u| \vee |v| \leq R$, $i \in \mathbb{S}$.

For any $i \in \mathbb{S}$, there exist constants $h_i \in \mathbb{R}$ and $h > 0$ such that

$$|u-v|^2|\sigma(u,i)-\sigma(v,i)|^2 - 2|(u-v)^T(\sigma(u,i)-\sigma(v,i))|^2 \leq h_i|u-v|^4,$$

$$|\sigma(u,i)-\sigma(v,i)|^2 \leq h|u-v|^2,$$

for any $u,\ v \in \mathbb{R}^n$.

Define

$$\beta_j = 2\alpha_j + h_j, \quad \beta = (\beta_1, \cdots, \beta_N)^T, \quad \lambda = |\mu\beta|. \tag{29}$$

Theorem (Theorem 7, Li, Ma, Yang, Y. (2018))

Suppose that (H1) and $\mu\beta < 0$ hold, then the solutions of the SDE with Markovian switching converge to a unique invariant measure $\pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ with some exponential rate $\xi > 0$ in the Wasserstein distance.

We can now define the BEM scheme for the SDEs with Markovian switching.

Let $X_0 = x$, $\Lambda_0 = i$, and define

$$X_{k+1} = X_k + b(X_{k+1}, \Lambda_k)\triangle + \sigma(X_k, \Lambda_k)\triangle W_k, \quad k \geq 0, \tag{30}$$

where $\triangle W_k = W(t_{k+1}) - W(t_k)$.

$\{(X_k, \Lambda_k)\}_{k\geq 0}$ is a time homogeneous Markov chain.

Let $\mathbf{P}^{\Delta}_{k\triangle}(x, i; \mathrm{d}u \times \{l\})$ be the transition probability of the pair $\left(X^{x,i}_k, \Lambda^i_k\right)$.

Lemma

Under the conditions of the Theorem, there exists a constant $\overline{\triangle}$ *such that the numerical solution of BEM scheme with any initial value* $(x, i) \in \mathbb{R}^n \times \mathbb{S}$ *satisfies*

$$\sup_{k \geq 0} \mathbb{E}|X_k|^p \leq C(1 + |x|^p) \tag{31}$$

for any $\triangle \in (0, \overline{\triangle})$ *and any* $p \in (0, p_0)$*, where*

$$p_0 = 1 \wedge \min_{j \in \mathbb{S}, 2\beta_j + \lambda > 0} \{-4q_{jj}/(2\beta_j + \lambda)\} \wedge \lambda/32h.$$

Lemma

Under the conditions of Theorem , it holds that

$$\mathbb{E}|X_k^{x,i} - X_k^{y,j}|^p \le C(1 + |x|^p + |y|^p)\mathrm{e}^{-\varsigma k \triangle} \tag{32}$$

for any $\triangle \in (0, \overline{\triangle})$ *and for any* $p \in (0, p_0)$, $(x, i), (y, j) \in \mathbb{R}^n \times \mathbb{S}$, $\overline{\triangle}$ *and* $\varsigma > 0$ *is a constant.*

Theorem (Theorem 8, Li, Ma, Yang, Y. (2018))

Under the conditions of **Theorem 7**, *there is a positive* $\triangle^*$ *sufficiently small such that for any* $\triangle \in (0, \triangle^*)$, *the solutions of the BEM method converge to a unique invariant measure* $\pi^{\Delta} \in \mathcal{P}(\mathbb{R}^n \times \mathbb{S})$ *with some exponential rate* $\xi_{\triangle} > 0$ *in the Wasserstein distance.*

Theorem (Theorem 9, Li, Ma, Yang, Y. (2018))

Under the conditions of **Theorem 8**, $\lim_{\triangle \to 0} W_p(\pi, \pi^{\triangle}) = 0$. *Furthermore, if the drift term satisfies the polynomial growth condition, that is,*

$$|b(x,i) - b(y,i)|^2 \leq c_i(1 + |x|^q + |y|^q)|x-y|^2, \quad \forall x, y \in \mathbb{R}^n, i \in \mathbb{S},$$

then $W_p(\pi, \pi^{\triangle}) \leq C\triangle^{\gamma}$ *for some* $\gamma \in (0, p/2)$, *where* c_i, q *are positive constants.*

Example III Consider SDE with $\Lambda(t)$ taking values in $\mathbb{S} = \{1, 2\}$ with generator $Q = \begin{pmatrix} -5 & 5 \\ 1 & -1 \end{pmatrix}$. The system is regarded as the Markovian switching between

$$\begin{cases} \mathrm{d}Y_1(t) = \left[2Y_1(t) - Y_1^3(t) - Y_1(t)Y_2^2(t)\right]\mathrm{d}t - 3\mathrm{d}W_1(t) + \mathrm{d}W_2(t), \\ \mathrm{d}Y_2(t) = \left[1 + Y_2(t) - Y_2^3(t) - Y_2(t)Y_1^2(t)\right]\mathrm{d}t + 4\mathrm{d}W_1(t), \end{cases} \tag{33}$$

and

$$\begin{cases} \mathrm{d}Y_1(t) &= \left(Y_1(t) - 2Y_1(t)\sqrt{Y_1^2(t) + Y_2^2(t)} + 1\right)\mathrm{d}t \\ &\quad + (2Y_1(t) - Y_2(t) + 2)\mathrm{d}W_1(t) + (Y_1(t) - Y_2(t))\mathrm{d}W_2(t), \\ \mathrm{d}Y_2(t) &= \left(0.5Y_2(t) - 2Y_2(t)\sqrt{Y_1^2(t) + Y_2^2(t)} + 2\right)\mathrm{d}t \\ &\quad + (Y_1(t) + 2Y_2(t))\mathrm{d}W_1(t) + (Y_1(t) + Y_2(t) - 4)\mathrm{d}W_2(t), \end{cases} \tag{34}$$

with the initial data $Y(0) = 1$, $r(0) = 1$, where $W(t) = (W_1(t), W_2(t))^T$ is a two-dimensional Brownian motion. Obviously, the diffusion coefficient σ is global Lipschitz continuous with Lipschitz constant $h = 7$. Note that the drift coefficient b is neither the global Lipschitz continuous nor the linear growth, but we can derive that

$$(u-v)^T(b(u,1)-b(v,1)) \leq 2|u-v|^2 - \frac{1}{4}(|u|-|v|)^4 \leq 2|u-v|^2,$$

and

$$(u-v)^T(b(u,2)-b(v,2)) \leq |u-v|^2 - 2(|u|+|v|)(|u|-|v|)^2 \leq |u-v|^2,$$

i.e. $\alpha_1 = 2$ and $\alpha_2 = 1$ for all $u, v \in \mathbb{R}^2$ in Assumption (H1).

We furthermore observe that

$$|u-v|^2|\sigma(u,i)-\sigma(v,i)|^2-2|(u-v)^T(\sigma(u,i)-\sigma(v,i))|^2 \le h_i|u-v|^4,$$

holds with $h_1=0$ and $h_2=-3$ for all $u,v\in\mathbb{R}^2$. Direct calculation leads to $\beta_1=2\alpha_1+h_1=4,\ \beta_2=2\alpha_2+h_2=-1$. The unique stationary distribution of $\Lambda(t)$ is $\mu=(\mu_1,\mu_2)=(1/6,5/6)$, then $\mu\beta=\mu_1\beta_1+\mu_2\beta_2=-1/6<0$. Then the exact solution $(Y(t),\Lambda(t))$ admits a unique invariant measure $\pi\in\mathcal{P}(\mathbb{R}^n\times\mathbb{S})$, and for a given stepsize $\triangle$ the numerical solution of BEM scheme has a unique invariant measure $\pi^{\triangle}\in\mathcal{P}(\mathbb{R}^n\times\mathbb{S})$ approximating π in the Wasserstein metric.

References

- Bakhtin, Y., Hurth, T., Mattingly, J. C., Regularity of invariant densities for 1D-systems with random switching. Nonlinearity, 28(2015), 3755–3787.
- Cloez, B., Hairer, M, Exponential ergodicity for Markov processes with random switching. Bernoulli, 21(2015), 505-536.
- Li, X., Mao, X., Mukama, D., Yuan, C., Delay feedback control for switching diffusion systems based on discrete-time observations. SIAM J. Control Optim. 58 (2020), 2900–2926.
- Mao, X., Yuan, C., Yin, G., Numerical method for stationary distribution of stochastic differential equations with Markovian switching, *J. Comput. Appl. Math.*, **174** (2005), 1–27.

References

- Nguyen, D. H. Yin, G., Stability of stochastic functional differential equations with regime-switching: analysis using Dupire's functional Itô formula. Potential Anal. 53 (2020), 247–265.
- Shao, J., Xi, F., Strong ergodicity of the regime-switching diffusion processes. Stochastic Process. Appl. 123 (2013), 3903–3918.
- Xi, F., Zu, Ch., Wu, F., On strong Feller property, exponential ergodicity and large deviations principle for stochastic damping Hamiltonian systems with state-dependent switching. J. Differential Equations 286 (2021), 856–891.
- Xi, F., Yin, G., Stability of Regime-Switching Jump Diffusions, *SIAM J. Control Optim.*, **48** (2010), 525–4549.

Thanks A Lot !