An Optimal Pairs Trading Selling Rule Under a Regime-Switching Model

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What Is Pairs Trading?

- Google Search: Pairs trading is a market-neutral trading strategy that matches a long position with a short position in a pair of highly correlated instruments such as two stocks, exchange-traded funds (ETFs), currencies, commodities or options.

What Is Pairs Trading?

- Key: Simultaneously trade a pair of stocks with opposite directions.
- How: When their prices diverge (e.g., one stock moves up while the other moves down), the pairs trade would be triggered:
  Buy the weaker stock and short the stronger one and bet on the eventual price convergence.

Why Pairs Trading?

- After all, we are not (neither our smart machines) that great forecasting market directions ...
- Investment strategies producing higher returns with smooth equity curve are highly desirable.
- Pairs trading is designed to address these issues and meet the needs.
- Major advantage: 'market neutral.'
  It can be profitable under any market (bull or bear) conditions.

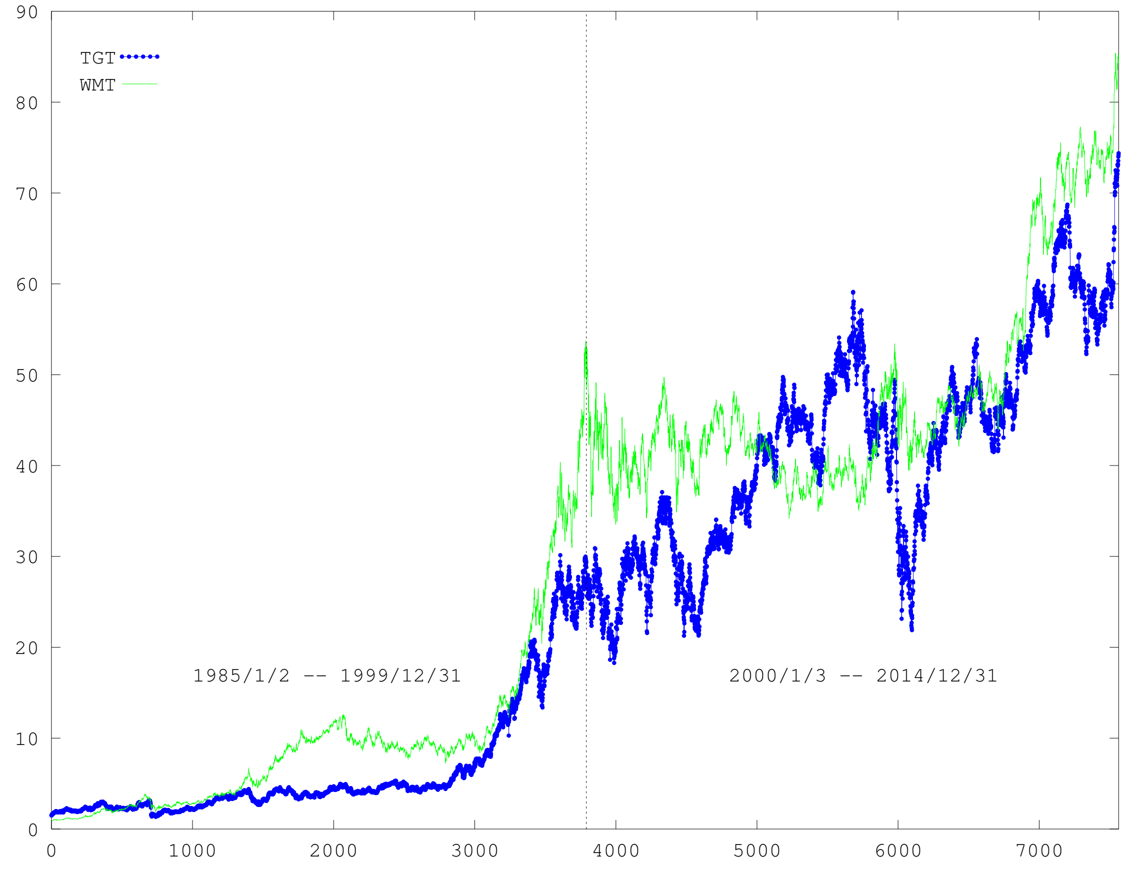
Implementation

It is important to determine when to initiate a pairs trade (i.e., how much divergence is sufficient to trigger a trade) and when to close the position (when to lock in profits).

Example.

We consider stock prices of Target Corp. (TGT) and Wal-Mart Stores Inc. (WMT).
Daily closing prices of both stocks from 1985 to 2014 are divided into two sections:
Part 1 (1985-1999) is used to calibrate the model (2-dim GBM);
Part 2 (2000-2014) can be used for backtesting.

Example: TGT and WMT Daily Closing Prices (1985 – 2014)



$$\mu_1 = 0.2059,\ \mu_2 = 0.2459,\ \sigma_{11} = 0.3112,\ \sigma_{12} = 0.0729,\ \sigma_{21} = 0.0729,\ \sigma_{22} = 0.2943.$$

Brief Background and Literature Review

Pairs Trading:

- Initially introduced by Bamberger and followed by Tartaglia's quantitative group at Morgan Stanley in the 1980s.
- Pairs trading (Gatev, Goetzmann, and Rouwenhorst, 2006)
- Book on pairs trading (Vidyamurthy, 2004)
- Pairs trading under a mean reversion model (Song and Zhang, 2013)
- Mean reversion trading (Zhang and Zhang, 2009 and Tie and Zhang 2016)
- Pairs trading under a two dimensional GBM (Tie, Zhang and Zhang, 2018)

McDonald and Siegel's Optimal Stopping

McDonald and Siegel in their 1986 paper (The value of waiting to invest) considered the optimal timing of investment in an irreversible project. Two main variables in their model: The value of the project X_t^1 and the cost of investing X_t^2.
Assuming both X_t^1 and X_t^2 are geometric Brownian motions, they demonstrated that one should defer the investment until the present value of the benefits X_t^1 from the project exceeds the investment cost X_t^2 by a certain margin.
Namely, invest only if $X_t^1 \geq k_0 X_t^2$ for some constant $k_0 > 1$ (or if $X_t^2 \leq k X_t^1$ for some $k < 1$).

Then in 1998, this problem was studied under precise optimality conditions by Hu and Øksendal. They also provided rigorous mathematical proofs.

The Connection ...

- McDonald and Siegel's problem can be easily interpreted in terms of pairs trading.
- It is a simple pairs trading selling rule!

  The project value = the long position price;
  The investment cost = the short position price

What's new?

- Consider a two-dimensional geometric Brownian motion with regime switching.
- Focus on when to close a pair position. Therefore, extend the McDonald and Siegel's result to models with regime switching.
- Obtain (nearly) closed-form solutions and establish their optimality.

Why regime switching?

- Market trends (bulls and bears)
- Fed interest rates
- Boom and bust cycles
- ... ...

The Model

- We consider two stocks $\mathbf{S}^1$ and $\mathbf{S}^2$. Let $\{X_t^1, t \geq 0\}$ denote the prices of stock $\mathbf{S}^1$ and $\{X_t^2, t \geq 0\}$ that of stock $\mathbf{S}^2$. They satisfy

$$d\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} X_t^1 & \\ & X_t^2 \end{pmatrix} \left[\begin{pmatrix} \mu_1(\alpha_t) \\ \mu_2(\alpha_t) \end{pmatrix} dt + \begin{pmatrix} \sigma_{11}(\alpha_t) & \sigma_{12}(\alpha_t) \\ \sigma_{21}(\alpha_t) & \sigma_{22}(\alpha_t) \end{pmatrix} d\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} \right],$$

  where μ_i, $i = 1, 2$, are the return rates, σ_{ij}, $i, j = 1, 2$, the volatility constants, α_t a two-state Markov chain, and (W_t^1, W_t^2) a 2-dimensional standard Brownian motion.
- $\alpha_t \in \mathcal{M} = \{1, 2\}$ is a Markov chain with generator $Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$ for positive λ_1 and λ_2.
- Assume α_t and (W_t^1, W_t^2) are independent.

The Model

- Consider a pairs trade selling rule. Assume the corresponding pair's position consists of a one-share long position in stock $\mathbf{S}^1$ and a one-share short position in stock $\mathbf{S}^2$.
- The problem is to determine an optimal stopping time τ to close the pair's position by selling $\mathbf{S}^1$ and buying back $\mathbf{S}^2$.
- Let K denote the transaction cost percentage (e.g., slippage and/or commission) associated with stock transactions.

  For example, the proceeds to close the pairs position at t is $(1-K)X_t^1 - (1+K)X_t^2$.

The Model

- Given the initial state (x_1, x_2), $\alpha = 1, 2$, and the selling time τ, the corresponding reward function

$$J(x_1, x_2, \alpha, \tau) = E\big[e^{-\rho\tau}(\beta_{\rm s} X^1_\tau - \beta_{\rm b} X^2_\tau)\big],$$

  where $\rho > 0$ is a given discount factor, $\beta_{\rm b} = 1 + K$, and $\beta_{\rm s} = 1 - K$.
- Let $\mathcal{F}_t = \sigma\{(X^1_r, X^2_r, \alpha_r) : \ r \le t\}$. The problem is to find an $\{\mathcal{F}_t\}$ stopping time τ to maximize J. Let $V(x_1, x_2, \alpha)$ denote the corresponding value functions:

$$V(x_1, x_2, \alpha) = \sup_{\tau} J(x_1, x_2, \alpha, \tau).$$

- The 'one-share' pair position is not as restrictive as it appears. For example, one can consider any pairs with n_1 shares of long position in $\mathbf{S}^1$ and n_2 shares of short position in $\mathbf{S}^2$. To treat this case, one only has to make change of the state variables $(X_t^1, X_t^2) \to (n_1 X_t^1, n_2 X_t^2)$. Due to the nature of GBMs, the corresponding system equation in will remain the same.
  The modification only affects the reward function implicitly.

The Model

- We impose the following conditions:
  (A1) For $\alpha = 1, 2$, $\rho > \mu_1(\alpha)$ and $\rho > \mu_2(\alpha)$.
- We have the lower and upper bounds for V:

$$\boxed{\beta_{\rm s} x_1 - \beta_{\rm b} x_2 \le V(x_1, x_2, \alpha) \le \beta_{\rm s} x_1}$$

Actually, the lower bound follows from the value function definition

$$V(x_1, x_2, \alpha) \ge J(x_1, x_2, \alpha, 0) = \beta_{\rm s} x_1 - \beta_{\rm b} x_2.$$

The upper bound can be obtained from Dynkin's formula

$$J(x_1, x_2, \alpha, \tau) \le E[e^{-\rho\tau} \beta_{\rm s} X_\tau^1] = \beta_{\rm s} \left(x_1 + E \int_0^\tau e^{-\rho t} X_t^1 (-\rho + \mu_1(\alpha_t)) dt \right) \le \beta_{\rm s} x_1.$$

HJB Equations

For $i = 1, 2$, let

$$\mathcal{A}_i = \frac{1}{2}\left[a_{11}(i)x_1^2\frac{\partial^2}{\partial x_1^2} + 2a_{12}(i)x_1x_2\frac{\partial^2}{\partial x_1 \partial x_2} + a_{22}(i)x_2^2\frac{\partial^2}{\partial x_2^2}\right] + \mu_1(i)x_1\frac{\partial}{\partial x_1} + \mu_2(i)x_2\frac{\partial}{\partial x_2}$$

where

$$a_{11}(i) = \sigma_{11}^2(i) + \sigma_{12}^2(i),\ a_{12}(i) = \sigma_{11}(i)\sigma_{21}(i) + \sigma_{12}(i)\sigma_{22}(i),\ \text{and } a_{22}(i) = \sigma_{21}^2(i) + \sigma_{22}^2(i).$$

Formally, the associated HJB equations have the form:

$$\begin{cases} \min\{(\rho - \mathcal{A}_1)v(x_1, x_2, 1) - \lambda_1(v(x_1, x_2, 2) - v(x_1, x_2, 1)), \\ \qquad\qquad v(x_1, x_2, 1) - \beta_{\mathrm{s}}x_1 + \beta_{\mathrm{b}}x_2\} = 0, \\ \min\{(\rho - \mathcal{A}_2)v(x_1, x_2, 2) - \lambda_2(v(x_1, x_2, 1) - v(x_1, x_2, 2)), \\ \qquad\qquad v(x_1, x_2, 2) - \beta_{\mathrm{s}}x_1 + \beta_{\mathrm{b}}x_2\} = 0. \end{cases}$$

To solve the HJB equations, we convert them into equations with a single variable $y = x_2/x_1$ and $v(x_1, x_2, i) = x_1 w_i(x_2/x_1)$, for some functions $w_i(y)$ and $i = 1, 2$.
Then it follows

$$\frac{\partial v(x_1,x_2,i)}{\partial x_1} = w_i(y) - y w_i'(y), \quad \frac{\partial v(x_1,x_2,i)}{\partial x_2} = w_i'(y),$$
$$\frac{\partial^2 v(x_1,x_2,i)}{\partial x_1^2} = \frac{y^2 w_i''(y)}{x_1}, \quad \frac{\partial^2 v(x_1,x_2,i)}{\partial x_2^2} = \frac{w_i''(y)}{x_1}, \text{ and } \frac{\partial^2 v(x_1,x_2,i)}{\partial x_1 \partial x_2} = -\frac{y w_i''(y)}{x_1}.$$

We rewrite $\mathcal{A}_i v(x_1, x_2, i)$ in terms of w_i:

$$\mathcal{A}_i v(x_1,x_2,i) = x_1 \left\{ \sigma_i y^2 w_i''(y) + [\mu_2(i) - \mu_1(i)] y w_i'(y) + \mu_1(i) w_i(y) \right\}.$$

where $\sigma_i = [a_{11}(i) - 2a_{12}(i) + a_{22}(i)]/2$.

Let

$$\mathcal{L}_i[w_i(y)] = \sigma_i y^2 w_i''(y) + [\mu_2(i) - \mu_1(i)] y w_i'(y) + \mu_1(i) w_i(y),\ \ i = 1, 2.$$

Then, the HJB equations can be given in terms of y and w_i as follows:

$$\begin{cases} \min\Big\{(\rho + \lambda_1 - \mathcal{L}_1) w_1(y) - \lambda_1 w_2(y),\ \ w_1(y) + \beta_{\rm b} y - \beta_{\rm s}\Big\} = 0, \\ \min\Big\{(\rho + \lambda_2 - \mathcal{L}_2) w_2(y) - \lambda_2 w_1(y),\ \ w_2(y) + \beta_{\rm b} y - \beta_{\rm s}\Big\} = 0. \end{cases}$$

Here, we only consider the case when $\sigma_i \neq 0$, $i = 1, 2.$ If either $\sigma_1 = 0$ and/or $\sigma_2 = 0$, the problem can be treated in a similar and much simpler way.

First we consider the equations:

$$(\rho + \lambda_1 - \mathcal{L}_1)w_1 = \lambda_1 w_2 \quad \text{and} \quad (\rho + \lambda_2 - \mathcal{L}_2)w_2 = \lambda_2 w_1.$$

Then both w_1 and w_2 satisfy the equation

$$[(\rho + \lambda_1 - \mathcal{L}_1)(\rho + \lambda_2 - \mathcal{L}_2) - \lambda_1\lambda_2]w = 0.$$

Note that both $\mathcal{L}_1$ and $\mathcal{L}_2$ are the classical Euler type operators and therefore the solutions to the above equation is of the form $w = y^\delta$ for some δ. Thus

$$[\rho + \lambda_1 - A_1(\delta)][\rho + \lambda_2 - A_2(\delta)] - \lambda_1\lambda_2 = 0,$$

with $A_i(\delta) = \sigma_i\delta(\delta - 1) + [(\mu_2(i) - \mu_1(i)]\delta + \mu_1(i),\ i = 1, 2.$
We can show the equation has four zeros $\delta_1 \geq \delta_2 > 1 > 0 > \delta_3 \geq \delta_4$.

Let $w_1 = \sum_{j=1}^{4} c_{1j} y^{\delta_j}$ and $w_2 = \sum_{j=1}^{4} c_{2j} y^{\delta_j}$, for some constants c_{ij}.
Then, it follows that

$$c_{1,j}(\rho + \lambda_1 - A_1(\delta_j)) = \lambda_1 c_{2j} \quad \text{and} \quad c_{2j}(\rho + \lambda_2 - A_2(\delta_j)) = \lambda_2 c_{1j}.$$

Define

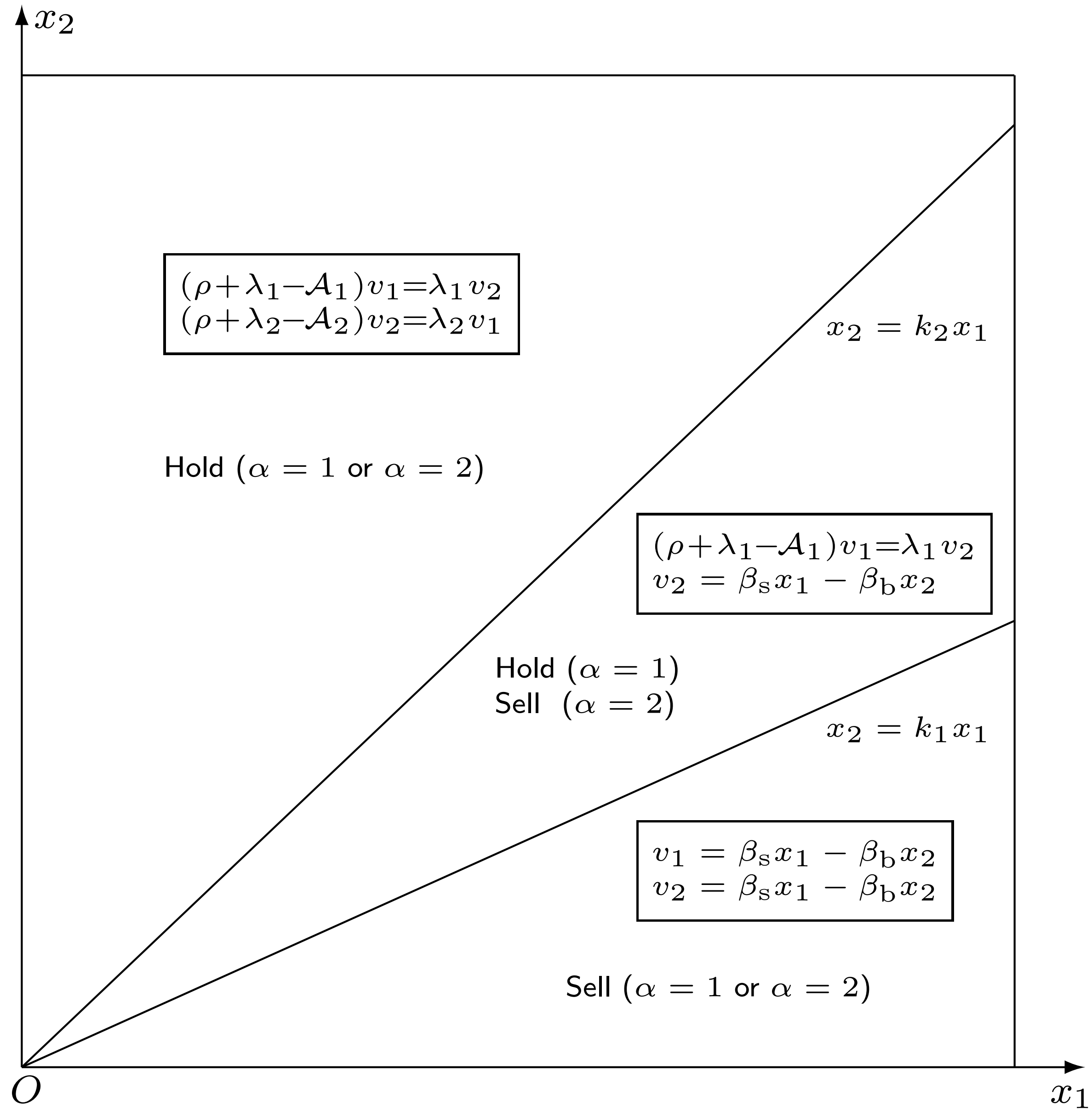
$$\eta_j = \frac{\rho + \lambda_1 - A_1(\delta_j)}{\lambda_1} \ \left(= \frac{\lambda_2}{\rho + \lambda_2 - A_2(\delta_j)} \right).$$

Therefore, $c_{2j} = \eta_j c_{1j}$, $j = 1, 2, 3, 4$. Hence,

$$w_1 = \sum_{j=1}^{4} c_{1j} y^{\delta_j} \quad \text{and} \quad w_2 = \sum_{j=1}^{4} \eta_j c_{1j} y^{\delta_j}.$$

Heuristically, one should close the pairs position when X_t^1 is large and X_t^2 is small. In view of this, we introduce $H_1 = \{(x_1, x_2) :\ x_2 \le k_1 x_1\}$ and $H_2 = \{(x_1, x_2) :\ x_2 \le k_2 x_1\}$, for some k_1 and k_2 so that one should sell when (X_t^1, X_t^2) enters H_i provided $\alpha_t = i$, $i = 1, 2$.
We consider two cases: $k_1 \le k_2$ and $k_2 \ge k_1$. By symmetry in $\alpha_t = 1$ and $\alpha_t = 2$, we only need to consider one of them, say $k_1 \le k_2$. We treat two separate cases: $k_1 < k_2$ and $k_1 = k_2$.

Regions for the Variational Inequalities



Case 1: $k_1 < k_2$

First, we divide $(0, \infty)$ into three intervals:

$$\Gamma_1 = (0, k_1], \quad \Gamma_2 = (k_1, k_2), \quad \text{and} \quad \Gamma_3 = [k_2, \infty).$$

Then, on each of these intervals, the HJB equations can be specified as follows:

$$\begin{cases} \Gamma_1: & w_1(y) = \beta_{\mathrm{s}} - \beta_{\mathrm{b}} y; & w_2(y) = \beta_{\mathrm{s}} - \beta_{\mathrm{b}} y; \\ \Gamma_2: & (\rho + \lambda_1 - \mathcal{L}_1) w_1(y) = \lambda_1 w_2(y); & w_2(y) = \beta_{\mathrm{s}} - \beta_{\mathrm{b}} y; \\ \Gamma_3: & (\rho + \lambda_1 - \mathcal{L}_1) w_1(y) = \lambda_1 w_2(y); & (\rho + \lambda_2 - \mathcal{L}_2) w_2(y) = \lambda_2 w_1(y). \end{cases}$$

We are to find solutions on each intervals. First, on Γ_3, recall the linear bounds for value functions and $\delta_1 > 1$ and $\delta_2 > 1$. It follows that the coefficients for y^{δ_1} and y^{δ_2} must be zero. Therefore,

$$w_1 = C_1 y^{\delta_3} + C_2 y^{\delta_4} \quad \text{and} \quad w_2 = \eta_3 C_1 y^{\delta_3} + \eta_4 C_2 y^{\delta_4}.$$

Next, to find solution on Γ_2, note that a particular solution for

$$(\rho + \lambda_1 - \mathcal{L}_1) w_1(y) = \lambda_1 w_2(y) = \lambda_1(\beta_{\mathrm{s}} - \beta_{\mathrm{b}} y)$$

can be given by $w_1 = a_1 + a_2 y$, with

$$a_1 = \frac{\lambda_1 \beta_{\mathrm{s}}}{\rho + \lambda_1 - \mu_1(1)} \quad \text{and} \quad a_2 = -\frac{\lambda_1 \beta_{\mathrm{b}}}{\rho + \lambda_1 - \mu_2(1)}.$$

To solve the above non-homogeneous equation, we only need to solve the homogeneous one $(\rho + \lambda_1 - \mathcal{L}_1)w_1 = 0$. Its solution is of the form y^γ. Then γ must be the roots of the quadratic equation

$$\sigma_1\gamma(\gamma - 1) + [(\mu_2(1) - \mu_1(1)]\gamma + \mu_1(1) - \rho - \lambda_1 = 0.$$

They are given by

$$\begin{cases} \gamma_1 = \dfrac{1}{2} + \dfrac{\mu_1(1) - \mu_2(1)}{2\sigma_1} + \sqrt{\left(\dfrac{1}{2} + \dfrac{\mu_1(1) - \mu_2(1)}{2\sigma_1}\right)^2 + \dfrac{\rho + \lambda_1 - \mu_1(1)}{\sigma_1}}, \\ \gamma_2 = \dfrac{1}{2} + \dfrac{\mu_1(1) - \mu_2(1)}{2\sigma_1} - \sqrt{\left(\dfrac{1}{2} + \dfrac{\mu_1(1) - \mu_2(1)}{2\sigma_1}\right)^2 + \dfrac{\rho + \lambda_1 - \mu_1(1)}{\sigma_1}} \end{cases}$$

The general solution for w_1 on Γ_2 is given by

$$w_1 = C_3 y^{\gamma_1} + C_4 y^{\gamma_2} + \frac{\lambda_1\beta_{\mathrm{s}}}{\rho + \lambda_1 - \mu_1(1)} - \frac{\lambda_1\beta_{\mathrm{b}}}{\rho + \lambda_1 - \mu_2(1)} y.$$

Smooth-fit conditions

We use smooth-fit conditions to set up equations for parameters k_1, k_2, C_1, C_2, C_3, and C_4.
First, the continuous differentiability of w_1 at k_1 yields

$$\begin{aligned} \beta_{\rm s} - \beta_{\rm b} k_1 = & \; C_3 k_1^{\gamma_1} + C_4 k_1^{\gamma_2} + a_1 + a_2 k_1, \\ -\beta_{\rm b} = & \; C_3 \gamma_1 k_1^{\gamma_1 - 1} + C_4 \gamma_2 k_1^{\gamma_2 - 1} + a_2. \end{aligned}$$

Similarly, we have the equation for w_2 at k_2

$$\begin{aligned} \beta_{\rm s} - \beta_{\rm b} k_2 = & \; \eta_3 C_1 k_2^{\delta_3} + \eta_4 C_2 k_2^{\delta_4}, \\ -\beta_{\rm b} = & \; \eta_3 \delta_3 C_1 k_2^{\delta_3 - 1} + \eta_4 \delta_4 C_2 k_2^{\delta_4 - 1}. \end{aligned}$$

Finally, the equations for w_1 at k_2 are given by

$$\begin{aligned} C_3 k_2^{\gamma_1} + C_4 k_2^{\gamma_2} + a_1 + a_2 k_2 = & \; C_1 k_2^{\delta_3} + C_2 k_2^{\delta_4}, \\ C_3 \gamma_1 k_2^{\gamma_1 - 1} + C_4 \gamma_2 k_2^{\gamma_2 - 1} + a_2 = & \; \delta_3 C_1 k_2^{\delta_3 - 1} + \delta_4 C_2 k_2^{\delta_4 - 1}. \end{aligned}$$

Using the first four equations, we solve for C_1, C_2, C_3, and C_4 in terms of k_1 and k_2

$$\begin{cases} C_1 = \dfrac{-\delta_4\beta_{\rm s} + (\delta_4 - 1)\beta_{\rm b} k_2}{\eta_3(\delta_3 - \delta_4)k_2^{\delta_3}}, & C_2 = \dfrac{\delta_3\beta_{\rm s} + (1 - \delta_3)\beta_{\rm b} k_2}{\eta_4(\delta_3 - \delta_4)k_2^{\delta_4}}, \\ C_3 = \dfrac{\gamma_2(\beta_{\rm s} - a_1) + (1 - \gamma_2)(\beta_{\rm b} + a_2)k_1}{(\gamma_2 - \gamma_1)k_1^{\gamma_1}}, & C_4 = \dfrac{-\gamma_1(\beta_{\rm s} - a_1) + (\gamma_1 - 1)(\beta_{\rm b} + a_2)k_1}{(\gamma_2 - \gamma_1)k_1^{\gamma_2}}. \end{cases}$$

Substitute these into the last two equations to obtain

$$\begin{aligned} &[\gamma_2(\beta_{\rm s} - a_1) + (1 - \gamma_2)(\beta_{\rm b} + a_2)k_1]\left(\frac{k_2}{k_1}\right)^{\gamma_1} + \gamma_2 a_1 + (\gamma_2 - 1)a_2 k_2 \\ &\quad = \frac{-\delta_4\beta_{\rm s} + (\delta_4 - 1)\beta_{\rm b} k_2}{\eta_3(\delta_3 - \delta_4)}(\gamma_2 - \delta_3) + \frac{\delta_3\beta_{\rm s} + (1 - \delta_3)\beta_{\rm b} k_2}{\eta_4(\delta_3 - \delta_4)}(\gamma_2 - \delta_4) \end{aligned}$$

and

$$\begin{aligned} &[-\gamma_1(\beta_{\rm s} - a_1) + (\gamma_1 - 1)(\beta_{\rm b} + a_2)k_1]\left(\frac{k_2}{k_1}\right)^{\gamma_2} + (1 - \gamma_1)a_2 k_2 - \gamma_1 a_1 \\ &\quad = \frac{-\delta_4\beta_{\rm s} + (\delta_4 - 1)\beta_{\rm b} k_2}{\eta_3(\delta_3 - \delta_4)}(\delta_3 - \gamma_1) + \frac{\delta_3\beta_{\rm s} + (1 - \delta_3)\beta_{\rm b} k_2}{\eta_4(\delta_3 - \delta_4)}(\delta_4 - \gamma_1). \end{aligned}$$

To reduce the above equations into linear equations in k_1 and k_2, we let $r = k_2/k_1$. Then, we have

$$\begin{cases} (\gamma_2(\beta_{\mathrm{s}} - a_1) + (1-\gamma_2)(\beta_{\mathrm{b}} + a_2)k_1)r^{\gamma_1} = A_1 + B_1 r k_1, \\ (-\gamma_1(\beta_{\mathrm{s}} - a_1) + (\gamma_1 - 1)(\beta_{\mathrm{b}} + a_2)k_1)r^{\gamma_2} = A_2 + B_2 r k_1. \end{cases}$$

where

$$\begin{cases} A_1 = \dfrac{-\delta_4\beta_{\mathrm{s}}(\gamma_2 - \delta_3)}{\eta_3(\delta_3 - \delta_4)} + \dfrac{\delta_3\beta_{\mathrm{s}}(\gamma_2 - \delta_4)}{\eta_4(\delta_3 - \delta_4)} - \gamma_2 a_1, \\ A_2 = \dfrac{-\delta_4\beta_{\mathrm{s}}(\delta_3 - \gamma_1)}{\eta_3(\delta_3 - \delta_4)} + \dfrac{\delta_3\beta_{\mathrm{s}}(\delta_4 - \gamma_1)}{\eta_4(\delta_3 - \delta_4)} + \gamma_1 a_1, \\ B_1 = \dfrac{(\delta_4 - 1)(\gamma_2 - \delta_3)\beta_{\mathrm{b}}}{\eta_3(\delta_3 - \delta_4)} + \dfrac{(1-\delta_3)\beta_{\mathrm{b}}(\gamma_2 - \delta_4)}{\eta_4(\delta_3 - \delta_4)} - (\gamma_2 - 1)a_2, \\ B_2 = \dfrac{(\delta_4 - 1)(\delta_3 - \gamma_1)\beta_{\mathrm{b}}}{\eta_3(\delta_3 - \delta_4)} + \dfrac{(1-\delta_3)\beta_{\mathrm{b}}(\delta_4 - \gamma_1)}{\eta_4(\delta_3 - \delta_4)} - (1-\gamma_1)a_2. \end{cases}$$

Eliminate k_1 to obtain the equation in r:

$$\frac{A_1 - \gamma_2(\beta_\mathrm{s} - a_1)r^{\gamma_1}}{(1-\gamma_2)(\beta_\mathrm{b} + a_2)r^{\gamma_1} - B_1 r} = \frac{A_2 + \gamma_1(\beta_\mathrm{s} - a_1)r^{\gamma_2}}{(\gamma_1 - 1)(\beta_\mathrm{b} + a_2)r^{\gamma_2} - B_2 r}$$

Let

$$f(r) = \frac{A_1 - \gamma_2(\beta_\mathrm{s} - a_1)r^{\gamma_1}}{(1-\gamma_2)(\beta_\mathrm{b} + a_2)r^{\gamma_1} - B_1 r} - \frac{A_2 + \gamma_1(\beta_\mathrm{s} - a_1)r^{\gamma_2}}{(\gamma_1 - 1)(\beta_\mathrm{b} + a_2)r^{\gamma_2} - B_2 r}.$$

We assume **(A2)**: $f(r)$ has a zero $r_0 > 1$.
Use this r_0 and recall $k_2 = r_0 k_1$ to obtain

$$\begin{cases} k_1 = \dfrac{A_1 - \gamma_2(\beta_\mathrm{s} - a_1)r_0^{\gamma_1}}{(1-\gamma_2)(\beta_\mathrm{b} + a_2)r_0^{\gamma_1} - B_1 r_0} = \dfrac{A_2 + \gamma_1(\beta_\mathrm{s} - a_1)r_0^{\gamma_2}}{(\gamma_1 - 1)(\beta_\mathrm{b} + a_2)r_0^{\gamma_2} - B_2 r_0}, \\ k_2 = r_0 k_1 = \dfrac{A_1 r_0 - \gamma_2(\beta_\mathrm{s} - a_1)r_0^{\gamma_1+1}}{(1-\gamma_2)(\beta_\mathrm{b} + a_2)r_0^{\gamma_1} - B_1 r_0} = \dfrac{A_2 r_0 + \gamma_1(\beta_\mathrm{s} - a_1)r_0^{\gamma_2+1}}{(\gamma_1 - 1)(\beta_\mathrm{b} + a_2)r_0^{\gamma_2} - B_2 r_0}. \end{cases}$$

Using these k_1 and k_2, we can express C_1, C_2, C_3, and C_4. Therefore, the solution w_1 and w_2 are given by

$$w_1(y) = \begin{cases} \beta_s - \beta_b y & \text{for } y \in \Gamma_1, \\ C_3 y^{\gamma_1} + C_4 y^{\gamma_2} + a_1 + a_2 y & \text{for } y \in \Gamma_2, \\ C_1 y^{\delta_3} + C_2 y^{\delta_4} & \text{for } y \in \Gamma_3; \end{cases}$$

$$w_2(y) = \begin{cases} \beta_s - \beta_b y & \text{for } y \in \Gamma_1 \cup \Gamma_2, \\ C_1 \eta_3 y^{\delta_3} + C_2 \eta_4 y^{\delta_4} & \text{for } y \in \Gamma_3. \end{cases}$$

Note that the variational inequalities in the HJB equations need to hold. In particular, we need the HJB inequalities to hold:

$$\begin{array}{lll} \Gamma_1: & (\rho + \lambda_1 - \mathcal{L}_1) w_1(y) - \lambda_1 w_2(y) \geq 0, & (\rho + \lambda_2 - \mathcal{L}_2) w_2(y) - \lambda_2 w_1(y) \geq 0; \\ \Gamma_2: & w_1 \geq \beta_s - \beta_b y, & (\rho + \lambda_2 - \mathcal{L}_2) w_2(y) - \lambda_2 w_1(y) \geq 0; \\ \Gamma_3: & w_1 \geq \beta_s - \beta_b y, & w_2 \geq \beta_s - \beta_b y. \end{array}$$

The inequalities on Γ_1 is equivalent to

$$\boxed{k_1 \leq \min\left\{\frac{(\rho-\mu_1(1))\beta_{\rm s}}{(\rho-\mu_2(1))\beta_{\rm b}}, \frac{(\rho-\mu_1(2))\beta_{\rm s}}{(\rho-\mu_2(2))\beta_{\rm b}}\right\}.}$$

Similarly, the second inequality in on Γ_2 are equivalent to

$$\boxed{w_1(y) \leq \beta_{\rm s} - \beta_{\rm b} y + \frac{1}{\lambda_2}[(\rho-\mu_1(2))\beta_{\rm s} - (\rho-\mu_2(2))\beta_{\rm b} y].}$$

Let $\phi(y) = w_1(y) - \beta_\mathrm{s} + \beta_\mathrm{b} y$. Then we can show the first inequality on Γ_2 is equivalent to

$$\begin{cases} \phi''(k_1) = C_3\gamma_1(\gamma_1 - 1)k_1^{\gamma_1 - 2} + C_4\gamma_2(\gamma_2 - 1)k_1^{\gamma_2 - 2} \geq 0 \text{ and} \\ \phi(k_2) = C_3 k_2^{\gamma_1} + C_4 k_2^{\gamma_2} + a_1 + a_2 k_2 - \beta_\mathrm{s} + \beta_\mathrm{b} y \geq 0. \end{cases}$$

Finally, let $\psi(y) = w_2(y) - \beta_\mathrm{s} + \beta_\mathrm{b} y$. Then, we can show the second inequality on Γ_3 is quivalent to

$$\psi''(k_2) = C_1\eta_3\delta_3(\delta_3 - 1)k_2^{\delta_3 - 2} + C_2\eta_4\delta_4(\delta_4 - 1)k_2^{\delta_4 - 2} \geq 0.$$

The other inequality on Γ_3 is equivalent to $\boxed{C_1 y^{\delta_3} + C_2 y^{\delta_4} \geq \beta_\mathrm{s} - \beta_\mathrm{b} y.}$
We assume **(A3)** The inequalities in blue boxes hold.

Case 2: $k_1 = k_2$

In the case, let $k_0 = k_1 = k_2$. We can show $k_0 = \dfrac{-\gamma_0 \beta_{\mathrm{s}}}{(1-\gamma_0)\beta_{\mathrm{b}}}$, where

$$\gamma_0 = \frac{1}{2} + \frac{\mu_1(1) - \mu_2(1)}{2\sigma_1} - \sqrt{\left(\frac{1}{2} + \frac{\mu_1(1) - \mu_2(1)}{2\sigma_1}\right)^2 + \frac{\rho - \mu_1(1)}{\sigma_1}} < 0.$$

Let

$$C_1 = \frac{\beta_{\mathrm{s}}^{1-\gamma_0} \beta_{\mathrm{b}}^{\gamma_0}}{(-\gamma_0)^{\gamma_0}(1-\gamma_0)^{1-\gamma_0}}.$$

Note that $\gamma_0 < 0$ which implies that both k_0 and C_1 are positive. The solution to the HJB equations is given by

$$w_1(y) = w_2(y) = w(y) = \begin{cases} \beta_{\mathrm{s}} - \beta_{\mathrm{b}} y & \text{for } y \in (0, k_0], \\ C_1 y^{\gamma_0} & \text{for } y \in (k_0, \infty). \end{cases}$$

In addition all variational inequalities hold.

A Verification Theorem

We provide a verification theorem for both Cases 1 and 2.
In Case 1, assume (A1), (A2), and (A3). In Case 2, assume (A1). Then, $v(x_1, x_2, \alpha) = x_1 w_\alpha(x_2/x_1) = V(x_1, x_2, \alpha)$, $\alpha = 1, 2$. Let $D = \{(x_1, x_2, 1) : \ x_2 > k_1 x_1\} \cup \{(x_1, x_2, 2) : \ x_2 > k_2 x_1\}$. Let $\tau^* = \inf\{t : \ (X_t^1, X_t^2, \alpha_t) \notin D\}$. Then τ^* is optimal.

Example: Case 1 ($k_1 < k_2$)

In this example, we take

$$\begin{array}{llll} \mu_1(1) = 0.20, & \mu_2(1) = 0.25, & \mu_1(2) = -0.30, & \mu_2(2) = -0.35, \\ \sigma_{11}(1) = 0.30, & \sigma_{12}(1) = 0.10, & \sigma_{21}(1) = 0.10, & \sigma_{22}(1) = 0.35, \\ \sigma_{11}(2) = 0.40, & \sigma_{12}(2) = 0.20, & \sigma_{21}(2) = 0.20, & \sigma_{22}(2) = 0.45, \\ \lambda_1 = 6.0, & \lambda_2 = 10.0, & K = 0.001, & \rho = 0.50. \end{array}$$

Then, the unique zero of $f(r)$ ($r > 1$) is given by $r_0 = 1.020254$. Using this r_0, we obtain $k_1 = 0.723270$ and $k_2 = 0.737920$. Then, we calculate and get $C_1 = 0.11442$, $C_2 = -0.00001$, $C_3 = 0.29121$, $C_4 = 0.00029$, $\eta_3 = 0.985919$, and $\eta_4 = -1.541271$. With these numbers, we verify all variational inequalities required in (A3).

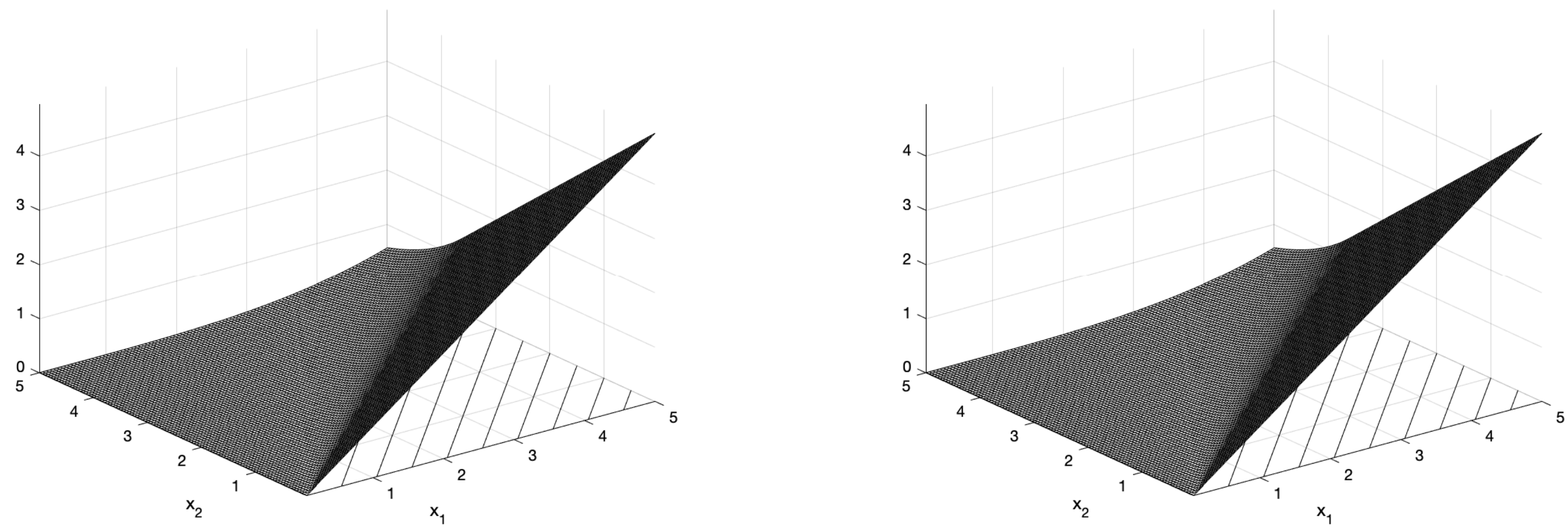


Figure 1: Value Functions $V(x_1, x_2, 1)$ and $V(x_1, x_2, 2)$

Example ($k_1 = k_2$)

In this example, we take

$$\begin{array}{ll} \mu_1(1) = \mu_1(2) = 0.20, & \mu_2(1) = \mu_2(2) = 0.25, \\ \sigma_{11}(1) = \sigma_{11}(2) = 0.30, & \sigma_{12}(1) = \sigma_{12}(2) = 0.10, \\ \sigma_{21}(1) = \sigma_{21}(2) = 0.10, & \sigma_{22}(1) = \sigma_{22}(2) = 0.35, \\ \lambda_1 = 6.0,\ \lambda_2 = 10.0, & K = 0.001,\ \rho = 0.50. \end{array}$$

We have $k_0 = 0.705098$ and $C_1 = 0.126431$. This gives the corresponding value function.

Example ($k_1 = k_2$)

In this example, we take

$$\begin{aligned} &\mu_1(1) = \mu_1(2) = 0.20, && \mu_2(1) = \mu_2(2) = 0.25, \\ &\sigma_{11}(1) = \sigma_{11}(2) = 0.30, && \sigma_{12}(1) = \sigma_{12}(2) = 0.10, \\ &\sigma_{21}(1) = \sigma_{21}(2) = 0.10, && \sigma_{22}(1) = \sigma_{22}(2) = 0.35, \\ &\lambda_1 = 6.0,\ \lambda_2 = 10.0, && K = 0.001,\ \rho = 0.50. \end{aligned}$$

We have $k_0 = 0.705098$ and $C_1 = 0.126431$. This gives the corresponding value function.

A sufficient and necessary condition for $k_1 = k_2$ is $\gamma_0|_{\alpha=1} = \gamma_0|_{\alpha=2}$.

Value Function $V(x_1, x_2) = V(x_1, x_2, 1) = V(x_1, x_2, 2)$

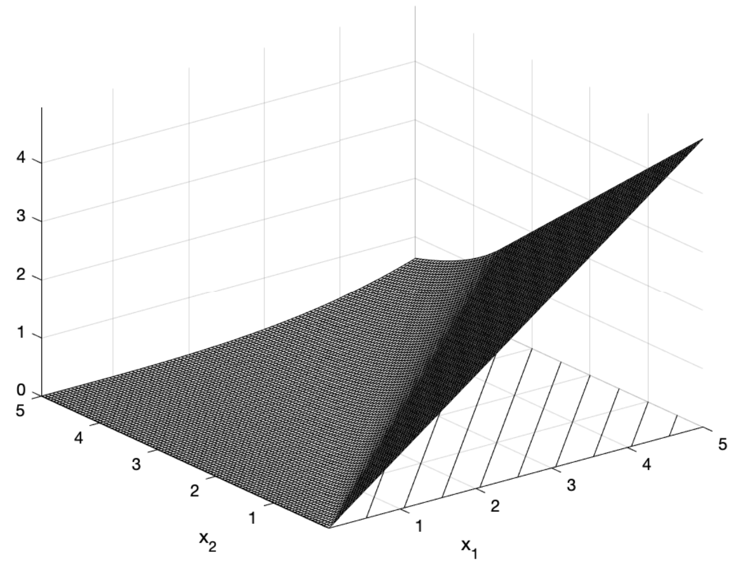


Figure 2: Value Function $V(x_1, x_2) = V(x_1, x_2, 1) = V(x_1, x_2, 2)$

Example ($k_1 > k_2$).

Finally, we take a different set of parameters:

$$\begin{array}{llll} \mu_1(1) = -0.10, & \mu_2(1) = 0.20, & \mu_1(2) = 0.25, & \mu_2(2) = -0.15, \\ \sigma_{11}(1) = 0.35, & \sigma_{12}(1) = 0.15, & \sigma_{21}(1) = 0.15, & \sigma_{22}(1) = 0.30, \\ \sigma_{11}(2) = 0.20, & \sigma_{12}(2) = 0.10, & \sigma_{21}(2) = 0.10, & \sigma_{22}(2) = 0.15, \\ \lambda_1 = 6.0, & \lambda_2 = 10.0, & K = 0.001, & \rho = 0.50. \end{array}$$

In this example, if we apply the same procedure used in Example 1 for k_1 and k_2, we notice some of the variational inequalities in (A3) will be violated. This means the condition $k_1 < k_2$ does not apply. Based on the symmetry of the problem in $\alpha = 1$ and $\alpha = 2$, we switch the set of parameters about $\alpha = 1$ and $\alpha = 2$ and obtain $\tilde{k}_1 = 0.379300$ and $\tilde{k}_2 = 0.824070$. The 'new' value functions ($\tilde{V}(x_1, x_2, 1)$, $\tilde{V}(x_1, x_2, 2)$) can be obtained in a similar way. So are the verification of the variational inequalities in (A3). Then, we switch back to obtain $k_1 = \tilde{k}_2 = 0.824070$ and $k_2 = \tilde{k}_1 = 0.379300$. The same for the value functions ($V(x_1, x_2, 1) = \tilde{V}(x_1, x_2, 2)$ and $V(x_1, x_2, 2) = \tilde{V}(x_1, x_2, 1)$).

Value Functions $V(x_1, x_2, 1)$ and $V(x_1, x_2, 2)$

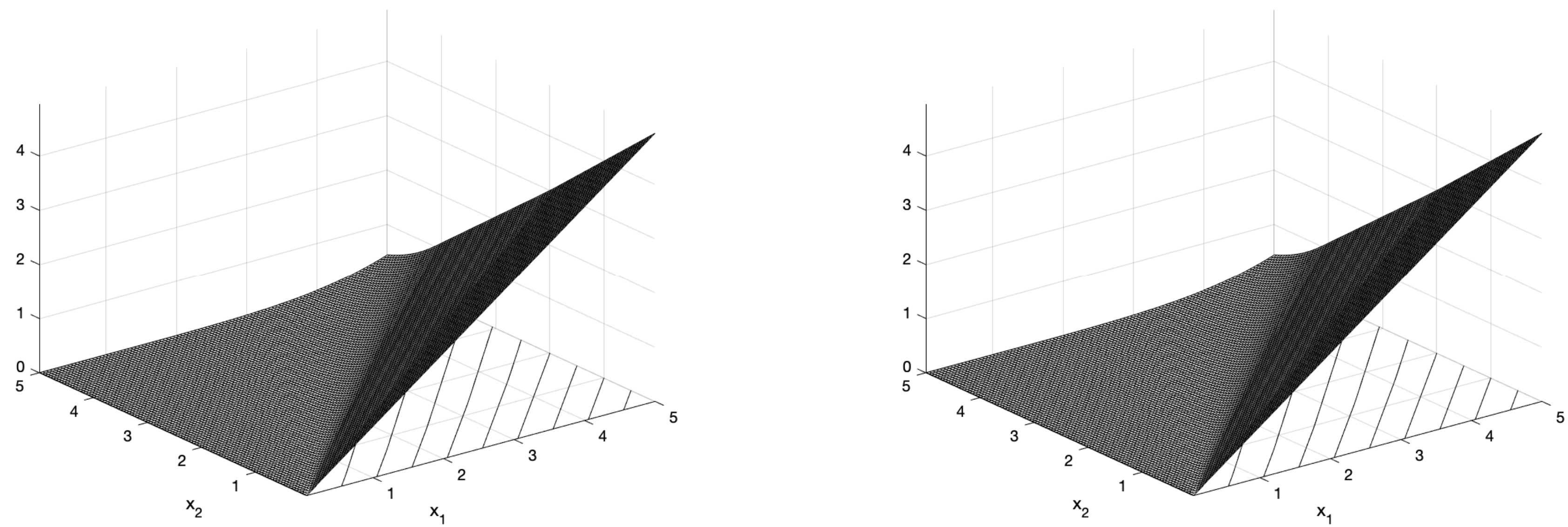


Figure 3: Value Functions $V(x_1, x_2, 1)$ and $V(x_1, x_2, 2)$

Conclusion

- The main focus of this paper is on a pairs trade selling rule.
- It extends the results of McDonald and Siegel (and Hu and Øksendal) by incorporating models with regime switching.
- It would be interesting to extend the results to include the buying side of optimal timing (in progress).
- It would also be interesting to consider models in which the market mode α_t is not directly observable.
  In this case, the Wonham filter can be used for calculation of the conditional probabilities of $\alpha = 1$ given the stock prices up to time t.

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