An Optimal Pairs Trading Selling Rule
Under a Regime-Switching Model

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Google Search: Pairs trading is a *market-neutral* trading strategy that matches a *long* position with a *short* position in a pair of *highly correlated* instruments such as two stocks, exchange-traded funds (ETFs), currencies, commodities or options.
What Is Pairs Trading?

- Key: Simultaneously trade a pair of stocks with opposite directions.
- How: When their prices diverge (e.g., one stock moves up while the other moves down), the pairs trade would be triggered: Buy the weaker stock and short the stronger one and bet on the eventual price convergence.
Why Pairs Trading?

- After all, we are not (neither our smart machines) that great forecasting market directions ...
- Investment strategies producing higher returns with smooth equity curve are highly desirable.
- Pairs trading is designed to address these issues and meet the needs.
- Major advantage: ‘market neutral.’
  It can be profitable under any market (bull or bear) conditions.
Implementation

It is important to determine when to initiate a pairs trade (i.e., how much divergence is sufficient to trigger a trade) and when to close the position (when to lock in profits).
We consider stock prices of Target Corp. (TGT) and Wal-Mart Stores Inc. (WMT).
Daily closing prices of both stocks from 1985 to 2014 are divided into two sections:
Part 1 (1985-1999) is used to calibrate the model (2-dim GBM);
Part 2 (2000-2014) can be used for backtesting.
\[ \mu_1 = 0.2059, \mu_2 = 0.2459, \sigma_{11} = 0.3112, \sigma_{12} = 0.0729, \sigma_{21} = 0.0729, \sigma_{22} = 0.2943. \]
Brief Background and Literature Review

Pairs Trading:

- Initially introduced by Bamberger and followed by Tartaglia’s quantitative group at Morgan Stanley in the 1980s.
- Pairs trading (Gatev, Goetzmann, and Rouwenhorst, 2006)
- Book on pairs trading (Vidyamurthy, 2004)
- Pairs trading under a mean reversion model (Song and Zhang, 2013)
- Mean reversion trading (Zhang and Zhang, 2009 and Tie and Zhang, 2016)
- Pairs trading under a two dimensional GBM (Tie, Zhang and Zhang, 2018)
McDonald and Siegel’s Optimal Stopping

McDonald and Siegel in their 1986 paper (The value of waiting to invest) considered the optimal timing of investment in an irreversible project. Two main variables in their model: The value of the project $X_t^1$ and the cost of investing $X_t^2$.

Assuming both $X_t^1$ and $X_t^2$ are geometric Brownian motions, they demonstrated that one should defer the investment until the present value of the benefits $X_t^1$ from the project exceeds the investment cost $X_t^2$ by a certain margin.

Namely, invest only if $X_t^1 \geq k_0 X_t^2$ for some constant $k_0 > 1$ (or if $X_t^2 \leq k X_t^1$ for some $k < 1$).

Then in 1998, this problem was studied under precise optimality conditions by Hu and Øksendal. They also provided rigorous mathematical proofs.
McDonald and Siegel’s problem can be easily interpreted in terms of pairs trading.

It is a simple pairs trading **selling** rule!

The project value = the long position price;
The investment cost = the short position price
What’s new?

- Consider a two-dimensional geometric Brownian motion with regime switching.
- Focus on when to close a pair position. Therefore, extend the McDonald and Siegel’s result to models with regime switching.
- Obtain (nearly) closed-form solutions and establish their optimality.
Why regime switching?

- Market trends (bulls and bears)
- Fed interest rates
- Boom and bust cycles
- ... ...
We consider two stocks $S^1$ and $S^2$. Let $\{X^1_t, t \geq 0\}$ denote the prices of stock $S^1$ and $\{X^2_t, t \geq 0\}$ that of stock $S^2$. They satisfy
\[
d\begin{pmatrix} X^1_t \\ X^2_t \end{pmatrix} = \begin{pmatrix} X^1_t \\ X^2_t \end{pmatrix} \left[ \begin{pmatrix} \mu_1(\alpha_t) \\ \mu_2(\alpha_t) \end{pmatrix} dt + \begin{pmatrix} \sigma_{11}(\alpha_t) & \sigma_{12}(\alpha_t) \\ \sigma_{21}(\alpha_t) & \sigma_{22}(\alpha_t) \end{pmatrix} \begin{pmatrix} W^1_t \\ W^2_t \end{pmatrix} \right],
\]
where $\mu_i, i = 1, 2$, are the return rates, $\sigma_{ij}, i, j = 1, 2$, the volatility constants, $\alpha_t$ a two-state Markov chain, and $(W^1_t, W^2_t)$ a 2-dimensional standard Brownian motion.

$\alpha_t \in \mathcal{M} = \{1, 2\}$ is a Markov chain with generator $Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$ for positive $\lambda_1$ and $\lambda_2$.

Assume $\alpha_t$ and $(W^1_t, W^2_t)$ are independent.
Consider a pairs trade selling rule. Assume the corresponding pair’s position consists of a one-share long position in stock $S^1$ and a one-share short position in stock $S^2$.

The problem is to determine an optimal stopping time $\tau$ to close the pair’s position by selling $S^1$ and buying back $S^2$.

Let $K$ denote the transaction cost percentage (e.g., slippage and/or commission) associated with stock transactions. For example, the proceeds to close the pairs position at $t$ is

$$(1 - K)X^1_t - (1 + K)X^2_t.$$
Given the initial state \((x_1, x_2), \alpha = 1, 2\), and the selling time \(\tau\), the corresponding reward function
\[
J(x_1, x_2, \alpha, \tau) = E\left[ e^{-\rho \tau} (\beta_s X_1^\tau - \beta_b X_2^\tau) \right],
\]
where \(\rho > 0\) is a given discount factor, \(\beta_b = 1 + K\), and \(\beta_s = 1 - K\).

Let \(\mathcal{F}_t = \sigma\{ (X_r^1, X_r^2, \alpha_r) : r \leq t \}\). The problem is to find an \(\{\mathcal{F}_t\}\) stopping time \(\tau\) to maximize \(J\). Let \(V(x_1, x_2, \alpha)\) denote the corresponding value functions:
\[
V(x_1, x_2, \alpha) = \sup_{\tau} J(x_1, x_2, \alpha, \tau).
\]
The ‘one-share’ pair position is not as restrictive as it appears. For example, one can consider any pairs with $n_1$ shares of long position in $S^1$ and $n_2$ shares of short position in $S^2$. To treat this case, one only has to make change of the state variables $(X^1_t, X^2_t) \rightarrow (n_1 X^1_t, n_2 X^2_t)$. Due to the nature of GBMs, the corresponding system equation in will remain the same.

The modification only affects the reward function implicitly.
The Model

- We impose the following conditions:
  - (A1) For \( \alpha = 1, 2, \rho > \mu_1(\alpha) \) and \( \rho > \mu_2(\alpha) \).
- We have the lower and upper bounds for \( V \):

\[
\beta_s x_1 - \beta_b x_2 \leq V(x_1, x_2, \alpha) \leq \beta_s x_1
\]

Actually, the lower bound follows from the value function definition

\[
V(x_1, x_2, \alpha) \geq J(x_1, x_2, \alpha, 0) = \beta_s x_1 - \beta_b x_2.
\]

The upper bound can be obtained from Dynkin’s formula

\[
J(x_1, x_2, \alpha, \tau) \leq E[e^{-\rho \tau} \beta_s X_1^\tau] = \beta_s \left( x_1 + E \int_0^\tau e^{-\rho t} X_1^t (-\rho + \mu_1(\alpha_t)) dt \right) \leq \beta_s x_1.
\]
For $i = 1, 2$, let
\[
A_i = \frac{1}{2} \left[ a_{11}(i)x_1^2 \frac{\partial^2}{\partial x_1^2} + 2a_{12}(i)x_1x_2 \frac{\partial^2}{\partial x_1 \partial x_2} + a_{22}(i)x_2^2 \frac{\partial^2}{\partial x_2^2} \right] + \mu_1(i)x_1 \frac{\partial}{\partial x_1} + \mu_2(i)x_2 \frac{\partial}{\partial x_2}
\]
where
\[
a_{11}(i) = \sigma_{11}^2(i), \quad a_{12}(i) = \sigma_{11}(i)\sigma_{21}(i) + \sigma_{12}(i)\sigma_{22}(i), \quad \text{and} \quad a_{22}(i) = \sigma_{21}^2(i) + \sigma_{22}^2(i).
\]
Formally, the associated HJB equations have the form:
\[
\begin{aligned}
\min \{(\rho - A_1) v(x_1, x_2, 1) - \lambda_1 (v(x_1, x_2, 2) - v(x_1, x_2, 1)), \\
v(x_1, x_2, 1) - \beta_s x_1 + \beta_b x_2 \} = 0,
\end{aligned}
\]
\[
\begin{aligned}
\min \{(\rho - A_2) v(x_1, x_2, 2) - \lambda_2 (v(x_1, x_2, 1) - v(x_1, x_2, 2)), \\
v(x_1, x_2, 2) - \beta_s x_1 + \beta_b x_2 \} = 0.
\end{aligned}
\]
To solve the HJB equations, we convert them into equations with a single variable $y = x_2/x_1$ and $v(x_1, x_2, i) = x_1 w_i(x_2/x_1)$, for some functions $w_i(y)$ and $i = 1, 2$.

Then it follows

$$
\frac{\partial v(x_1, x_2, i)}{\partial x_1} = w_i(y) - y w'_i(y), \quad \frac{\partial v(x_1, x_2, i)}{\partial x_2} = w'_i(y),
$$

$$
\frac{\partial^2 v(x_1, x_2, i)}{\partial x_1^2} = \frac{y^2 w''_i(y)}{x_1}, \quad \frac{\partial^2 v(x_1, x_2, i)}{\partial x_2^2} = \frac{w''_i(y)}{x_1}, \text{ and } \frac{\partial^2 v(x_1, x_2, i)}{\partial x_1 \partial x_2} = -\frac{y w''_i(y)}{x_1}.
$$

We rewrite $A_i v(x_1, x_2, i)$ in terms of $w_i$:

$$
A_i v(x_1, x_2, i) = x_1 \left\{ \sigma_i y^2 w''_i(y) + [\mu_2(i) - \mu_1(i)] y w'_i(y) + \mu_1(i) w_i(y) \right\}.
$$

where $\sigma_i = [a_{11}(i) - 2a_{12}(i) + a_{22}(i)]/2$. 
Let
\[
L_i[w_i(y)] = \sigma_i y^2 w_i''(y) + [\mu_2(i) - \mu_1(i)] y w_i'(y) + \mu_1(i) w_i(y), \quad i = 1, 2.
\]
Then, the HJB equations can be given in terms of \( y \) and \( w_i \) as follows:
\[
\begin{cases}
\min \left\{ (\rho + \lambda_1 - L_1) w_1(y) - \lambda_1 w_2(y), \ w_1(y) + \beta_b y - \beta_s \right\} = 0, \\
\min \left\{ (\rho + \lambda_2 - L_2) w_2(y) - \lambda_2 w_1(y), \ w_2(y) + \beta_b y - \beta_s \right\} = 0.
\end{cases}
\]
Here, we only consider the case when \( \sigma_i \neq 0, \ i = 1, 2 \). If either \( \sigma_1 = 0 \) and/or \( \sigma_2 = 0 \), the problem can be treated in a similar and much simpler way.
First we consider the equations:

\[(\rho + \lambda_1 - L_1)w_1 = \lambda_1 w_2 \quad \text{and} \quad (\rho + \lambda_2 - L_2)w_2 = \lambda_2 w_1.\]

Then both \(w_1\) and \(w_2\) satisfy the equation

\[\left[(\rho + \lambda_1 - L_1)(\rho + \lambda_2 - L_2) - \lambda_1 \lambda_2\right]w = 0.\]

Note that both \(L_1\) and \(L_2\) are the classical Euler type operators and therefore the solutions to the above equation is of the form \(w = y^\delta\) for some \(\delta\). Thus

\[\left[\rho + \lambda_1 - A_1(\delta)\right]\left[\rho + \lambda_2 - A_2(\delta)\right] - \lambda_1 \lambda_2 = 0,\]

with \(A_i(\delta) = \sigma_i \delta (\delta - 1) + [(\mu_2(i) - \mu_1(i))\delta + \mu_1(i)], \quad i = 1, 2.\)

We can show the equation has four zeros \(\delta_1 \geq \delta_2 > 1 > 0 > \delta_3 \geq \delta_4.\)
Let $w_1 = \sum_{j=1}^{4} c_{1j} y^{\delta_j}$ and $w_2 = \sum_{j=1}^{4} c_{2j} y^{\delta_j}$, for some constants $c_{ij}$.

Then, it follows that

$$c_{1j}(\rho + \lambda_1 - A_1(\delta_j)) = \lambda_1 c_{2j} \quad \text{and} \quad c_{2j}(\rho + \lambda_2 - A_2(\delta_j)) = \lambda_2 c_{1j}.$$ 

Define

$$\eta_j = \frac{\rho + \lambda_1 - A_1(\delta_j)}{\lambda_1} \left( = \frac{\lambda_2}{\rho + \lambda_2 - A_2(\delta_j)} \right).$$

Therefore, $c_{2j} = \eta_j c_{1j}, \ j = 1, 2, 3, 4$. Hence,

$$w_1 = \sum_{j=1}^{4} c_{1j} y^{\delta_j} \quad \text{and} \quad w_2 = \sum_{j=1}^{4} \eta_j c_{1j} y^{\delta_j}.$$
Heuristically, one should close the pairs position when $X^1_t$ is large and $X^2_t$ is small. In view of this, we introduce $H_1 = \{(x_1, x_2) : x_2 \leq k_1 x_1\}$ and $H_2 = \{(x_1, x_2) : x_2 \leq k_2 x_1\}$, for some $k_1$ and $k_2$ so that one should sell when $(X^1_t, X^2_t)$ enters $H_i$ provided $\alpha_t = i$, $i = 1, 2$.

We consider two cases: $k_1 \leq k_2$ and $k_2 \geq k_1$. By symmetry in $\alpha_t = 1$ and $\alpha_t = 2$, we only need to consider one of them, say $k_1 \leq k_2$. We treat two separate cases: $k_1 < k_2$ and $k_1 = k_2$. 
Regions for the Variational Inequalities

\[\begin{align*}
(\rho + \lambda_1 - \mathcal{A}_1)v_1 &= \lambda_1 v_2 \\
(\rho + \lambda_2 - \mathcal{A}_2)v_2 &= \lambda_2 v_1
\end{align*}\]

\[\begin{align*}
v_1 &= \beta_s x_1 - \beta_b x_2 \\
v_2 &= \beta_s x_1 - \beta_b x_2
\end{align*}\]

\[x_2 = k_2 x_1\]

\[x_2 = k_1 x_1\]

Hold (\(\alpha = 1\) or \(\alpha = 2\))

Sell (\(\alpha = 2\))

Hold (\(\alpha = 1\))

Sell (\(\alpha = 1\) or \(\alpha = 2\))
**Case 1:** $k_1 < k_2$

First, we divide $(0, \infty)$ into three intervals:

$\Gamma_1 = (0, k_1], \quad \Gamma_2 = (k_1, k_2), \quad \text{and} \quad \Gamma_3 = [k_2, \infty).$

Then, on each of these intervals, the HJB equations can be specified as follows:

\[
\begin{align*}
\Gamma_1 & : \quad w_1(y) = \beta_s - \beta_b y; \quad w_2(y) = \beta_s - \beta_b y; \\
\Gamma_2 & : \quad (\rho + \lambda_1 - L_1)w_1(y) = \lambda_1 w_2(y); \quad w_2(y) = \beta_s - \beta_b y; \\
\Gamma_3 & : \quad (\rho + \lambda_1 - L_1)w_1(y) = \lambda_1 w_2(y); \quad (\rho + \lambda_2 - L_2)w_2(y) = \lambda_2 w_1(y).
\end{align*}
\]
We are to find solutions on each intervals. First, on $\Gamma_3$, recall the linear bounds for value functions and $\delta_1 > 1$ and $\delta_2 > 1$. It follows that the coefficients for $y^{\delta_1}$ and $y^{\delta_2}$ must be zero. Therefore,

$$w_1 = C_1 y^{\delta_3} + C_2 y^{\delta_4} \quad \text{and} \quad w_2 = \eta_3 C_1 y^{\delta_3} + \eta_4 C_2 y^{\delta_4}.$$ 

Next, to find solution on $\Gamma_2$, note that a particular solution for

$$(\rho + \lambda_1 - \mathcal{L}_1)w_1(y) = \lambda_1 w_2(y) = \lambda_1 (\beta_s - \beta_b y)$$

can be given by $w_1 = a_1 + a_2 y$, with

$$a_1 = \frac{\lambda_1 \beta_s}{\rho + \lambda_1 - \mu_1(1)} \quad \text{and} \quad a_2 = -\frac{\lambda_1 \beta_b}{\rho + \lambda_1 - \mu_2(1)}.$$
To solve the above non-homogeneous equation, we only need to solve the homogeneous one \((\rho + \lambda_1 - L_1)w_1 = 0\). Its solution is of the form \(y^\gamma\). Then \(\gamma\) must be the roots of the quadratic equation

\[
\sigma_1 \gamma (\gamma - 1) + [(\mu_2(1) - \mu_1(1)]\gamma + \mu_1(1) - \rho - \lambda_1 = 0.
\]

They are given by

\[
\begin{align*}
\gamma_1 &= \frac{1}{2} + \frac{\mu_1(1) - \mu_2(1)}{2\sigma_1} + \sqrt{\left(\frac{1}{2} + \frac{\mu_1(1) - \mu_2(1)}{2\sigma_1}\right)^2 + \frac{\rho + \lambda_1 - \mu_1(1)}{\sigma_1}}, \\
\gamma_2 &= \frac{1}{2} + \frac{\mu_1(1) - \mu_2(1)}{2\sigma_1} - \sqrt{\left(\frac{1}{2} + \frac{\mu_1(1) - \mu_2(1)}{2\sigma_1}\right)^2 + \frac{\rho + \lambda_1 - \mu_1(1)}{\sigma_1}}.
\end{align*}
\]

The general solution for \(w_1\) on \(\Gamma_2\) is given by

\[
w_1 = C_3y^{\gamma_1} + C_4y^{\gamma_2} + \frac{\lambda_1\beta_s}{\rho + \lambda_1 - \mu_1(1)} - \frac{\lambda_1\beta_b}{\rho + \lambda_1 - \mu_2(1)}y.
\]
Smooth-fit conditions

We use smooth-fit conditions to set up equations for parameters $k_1$, $k_2$, $C_1$, $C_2$, $C_3$, and $C_4$.

First, the continuous differentiability of $w_1$ at $k_1$ yields

$$
\beta_s - \beta_b k_1 = C_3 k_1^{\gamma_1} + C_4 k_1^{\gamma_2} + a_1 + a_2 k_1,
$$

$$
-\beta_b = C_3 \gamma_1 k_1^{\gamma_1-1} + C_4 \gamma_2 k_1^{\gamma_2-1} + a_2.
$$

Similarly, we have the equation for $w_2$ at $k_2$

$$
\beta_s - \beta_b k_2 = \eta_3 C_1 k_2^{\delta_3} + \eta_4 C_2 k_2^{\delta_4},
$$

$$
-\beta_b = \eta_3 \delta_3 C_1 k_2^{\delta_3-1} + \eta_4 \delta_4 C_2 k_2^{\delta_4-1}.
$$

Finally, the equations for $w_1$ at $k_2$ are given by

$$
C_3 k_2^{\gamma_1} + C_4 k_2^{\gamma_2} + a_1 + a_2 k_2 = C_1 k_2^{\delta_3} + C_2 k_2^{\delta_4},
$$

$$
C_3 \gamma_1 k_2^{\gamma_1-1} + C_4 \gamma_2 k_2^{\gamma_2-1} + a_2 = \delta_3 C_1 k_2^{\delta_3-1} + \delta_4 C_2 k_2^{\delta_4-1}.$$
Using the first four equations, we solve for $C_1$, $C_2$, $C_3$, and $C_4$ in terms of $k_1$ and $k_2$

\[
\begin{align*}
C_1 &= \frac{-\delta_4 \beta_s + (\delta_4 - 1) \beta_b k_2}{\eta_3 (\delta_3 - \delta_4) k_2^{\delta_3}}, \\
C_3 &= \frac{\gamma_2 (\beta_s - a_1) + (1 - \gamma_2) (\beta_b + a_2) k_1}{(\gamma_2 - \gamma_1) k_1^{\gamma_1}} , \quad C_2 = \frac{\delta_3 \beta_s + (1 - \delta_3) \beta_b k_2}{\eta_4 (\delta_3 - \delta_4) k_2^{\delta_4}}, \\
\text{and} \quad C_4 &= \frac{-\gamma_1 (\beta_s - a_1) + (\gamma_1 - 1) (\beta_b + a_2) k_1}{(\gamma_2 - \gamma_1) k_1^{\gamma_2}}.
\end{align*}
\]

Substitute these into the last two equations to obtain

\[
[\gamma_2 (\beta_s - a_1) + (1 - \gamma_2) (\beta_b + a_2) k_1] \left( \frac{k_2}{k_1} \right)^{\gamma_1} + \gamma_2 a_1 + (\gamma_2 - 1) a_2 k_2
\]

\[
= \frac{-\delta_4 \beta_s + (\delta_4 - 1) \beta_b k_2}{\eta_3 (\delta_3 - \delta_4)} (\gamma_2 - \delta_3) + \frac{\delta_3 \beta_s + (1 - \delta_3) \beta_b k_2}{\eta_4 (\delta_3 - \delta_4)} (\gamma_2 - \delta_4)
\]

and

\[
[-\gamma_1 (\beta_s - a_1) + (\gamma_1 - 1) (\beta_b + a_2) k_1] \left( \frac{k_2}{k_1} \right)^{\gamma_2} + (1 - \gamma_1) a_2 k_2 - \gamma_1 a_1
\]

\[
= \frac{-\delta_4 \beta_s + (\delta_4 - 1) \beta_b k_2}{\eta_3 (\delta_3 - \delta_4)} (\delta_3 - \gamma_1) + \frac{\delta_3 \beta_s + (1 - \delta_3) \beta_b k_2}{\eta_4 (\delta_3 - \delta_4)} (\delta_4 - \gamma_1).
\]
To reduce the above equations into linear equations in $k_1$ and $k_2$, we let $r = k_2/k_1$. Then, we have

\[
\begin{align*}
\left\{ \begin{array}{l}
(\gamma_2(\beta_s - a_1) + (1 - \gamma_2)(\beta_b + a_2)k_1)r^{\gamma_1} = A_1 + B_1rk_1, \\
(-\gamma_1(\beta_s - a_1) + (\gamma_1 - 1)(\beta_b + a_2)k_1)r^{\gamma_2} = A_2 + B_2rk_1.
\end{array} \right.
\]

where

\[
\begin{align*}
A_1 &= \frac{-\delta_4\beta_s(\gamma_2 - \delta_3)}{\eta_3(\delta_3 - \delta_4)} + \frac{\delta_3\beta_s(\gamma_2 - \delta_4)}{\eta_4(\delta_3 - \delta_4)} - \gamma_2a_1, \\
A_2 &= \frac{-\delta_4\beta_s(\delta_3 - \gamma_1)}{\eta_3(\delta_3 - \delta_4)} + \frac{\delta_3\beta_s(\delta_4 - \gamma_1)}{\eta_4(\delta_3 - \delta_4)} + \gamma_1a_1, \\
B_1 &= \frac{(\delta_4 - 1)(\gamma_2 - \delta_3)\beta_b}{\eta_3(\delta_3 - \delta_4)} + \frac{(1 - \delta_3)\beta_b(\gamma_2 - \delta_4)}{\eta_4(\delta_3 - \delta_4)} - (\gamma_2 - 1)a_2, \\
B_2 &= \frac{(\delta_4 - 1)(\delta_3 - \gamma_1)\beta_b}{\eta_3(\delta_3 - \delta_4)} + \frac{(1 - \delta_3)\beta_b(\delta_4 - \gamma_1)}{\eta_4(\delta_3 - \delta_4)} - (1 - \gamma_1)a_2.
\end{align*}
\]
Eliminate \( k_1 \) to obtain the equation in \( r \):

\[
\frac{A_1 - \gamma_2(\beta_s - a_1)r^\gamma_1}{(1 - \gamma_2)(\beta_b + a_2)r^\gamma_1 - B_1r} = \frac{A_2 + \gamma_1(\beta_s - a_1)r^\gamma_2}{(\gamma_1 - 1)(\beta_b + a_2)r^\gamma_2 - B_2r}
\]

Let

\[
f(r) = \frac{A_1 - \gamma_2(\beta_s - a_1)r^\gamma_1}{(1 - \gamma_2)(\beta_b + a_2)r^\gamma_1 - B_1r} - \frac{A_2 + \gamma_1(\beta_s - a_1)r^\gamma_2}{(\gamma_1 - 1)(\beta_b + a_2)r^\gamma_2 - B_2r}.
\]

We assume (A2): \( f(r) \) has a zero \( r_0 > 1 \).

Use this \( r_0 \) and recall \( k_2 = r_0 k_1 \) to obtain

\[
\begin{align*}
k_1 &= \frac{A_1 - \gamma_2(\beta_s - a_1)r_0^{\gamma_1}}{(1 - \gamma_2)(\beta_b + a_2)r_0^{\gamma_1} - B_1r_0} = \frac{A_2 + \gamma_1(\beta_s - a_1)r_0^{\gamma_2}}{(\gamma_1 - 1)(\beta_b + a_2)r_0^{\gamma_2} - B_2r_0}, \\
k_2 &= r_0 k_1 = \frac{A_1 r_0 - \gamma_2(\beta_s - a_1)r_0^{\gamma_1+1}}{(1 - \gamma_2)(\beta_b + a_2)r_0^{\gamma_1} - B_1r_0} = \frac{A_2 r_0 + \gamma_1(\beta_s - a_1)r_0^{\gamma_2+1}}{(\gamma_1 - 1)(\beta_b + a_2)r_0^{\gamma_2} - B_2r_0}.
\end{align*}
\]
Using these $k_1$ and $k_2$, we can express $C_1$, $C_2$, $C_3$, and $C_4$. Therefore, the solution $w_1$ and $w_2$ are given by

\[
\begin{align*}
  w_1(y) &= \left\{ \begin{array}{ll}
    \beta_s - \beta_b y & \text{for } y \in \Gamma_1, \\
    C_3 y^{\gamma_1} + C_4 y^{\gamma_2} + a_1 + a_2 y & \text{for } y \in \Gamma_2, \\
    C_1 y^{\delta_3} + C_2 y^{\delta_4} & \text{for } y \in \Gamma_3;
  \end{array} \right.
  \\
  w_2(y) &= \left\{ \begin{array}{ll}
    \beta_s - \beta_b y & \text{for } y \in \Gamma_1 \cup \Gamma_2, \\
    C_1 \eta_3 y^{\delta_3} + C_2 \eta_4 y^{\delta_4} & \text{for } y \in \Gamma_3.
  \end{array} \right.
\end{align*}
\]

Note that the variational inequalities in the HJB equations need to hold. In particular, we need the HJB inequalities to hold:

\[
\begin{align*}
  \Gamma_1 : & \quad (\rho + \lambda_1 - \mathcal{L}_1) w_1(y) - \lambda_1 w_2(y) \geq 0, \\
  \Gamma_2 : & \quad w_1 \geq \beta_s - \beta_b y, \\
  \Gamma_3 : & \quad w_1 \geq \beta_s - \beta_b y,
\end{align*}
\]

\[
\begin{align*}
  (\rho + \lambda_2 - \mathcal{L}_2) w_2(y) - \lambda_2 w_1(y) & \geq 0; \\
  (\rho + \lambda_2 - \mathcal{L}_2) w_2(y) - \lambda_2 w_1(y) & \geq 0; \\
  w_2 & \geq \beta_s - \beta_b y.
\end{align*}
\]
The inequalities on $\Gamma_1$ is equivalent to

$$k_1 \leq \min \left\{ \frac{(\rho - \mu_1(1))\beta_s}{(\rho - \mu_2(1))\beta_b}, \frac{(\rho - \mu_1(2))\beta_s}{(\rho - \mu_2(2))\beta_b} \right\}.$$  

Similarly, the second inequality in on $\Gamma_2$ are equivalent to

$$w_1(y) \leq \beta_s - \beta_b y + \frac{1}{\lambda_2} \left[ (\rho - \mu_1(2))\beta_s - (\rho - \mu_2(2))\beta_b y \right].$$
Let $\phi(y) = w_1(y) - \beta_s + \beta_b y$. Then we can show the first inequality on $\Gamma_2$ is equivalent to

$$\begin{cases} 
\phi''(k_1) = C_3 \gamma_1 (\gamma_1 - 1) k_1^{\gamma_1-2} + C_4 \gamma_2 (\gamma_2 - 1) k_1^{\gamma_2-2} \geq 0 \text{ and } \\
\phi(k_2) = C_3 k_2^{\gamma_1} + C_4 k_2^{\gamma_2} + a_1 + a_2 k_2 - \beta_s + \beta_b y \geq 0.
\end{cases}$$

Finally, let $\psi(y) = w_2(y) - \beta_s + \beta_b y$. Then, we can show the second inequality on $\Gamma_3$ is equivalent to

$$\psi''(k_2) = C_1 \eta_3 \delta_3 (\delta_3 - 1) k_2^{\delta_3-2} + C_2 \eta_4 \delta_4 (\delta_4 - 1) k_2^{\delta_4-2} \geq 0.$$ 

The other inequality on $\Gamma_3$ is equivalent to

$$C_1 y^{\delta_3} + C_2 y^{\delta_4} \geq \beta_s - \beta_b y.$$ 

We assume \textbf{(A3)} The inequalities in blue boxes hold.
Case 2: \( k_1 = k_2 \)

In the case, let \( k_0 = k_1 = k_2 \). We can show \( k_0 = \frac{-\gamma_0 \beta_s}{(1 - \gamma_0) \beta_b} \), where

\[
\gamma_0 = \frac{1}{2} + \frac{\mu_1(1) - \mu_2(1)}{2\sigma_1} - \sqrt{\left(\frac{1}{2} + \frac{\mu_1(1) - \mu_2(1)}{2\sigma_1}\right)^2 + \frac{\rho - \mu_1(1)}{\sigma_1}} < 0.
\]

Let

\[
C_1 = \frac{\beta_s^{1-\gamma_0} \beta_b^{\gamma_0}}{(-\gamma_0)^{\gamma_0} (1 - \gamma_0)^{1-\gamma_0}}.
\]

Note that \( \gamma_0 < 0 \) which implies that both \( k_0 \) and \( C_1 \) are positive.

The solution to the HJB equations is given by

\[
w_1(y) = w_2(y) = w(y) = \begin{cases} 
\beta_s - \beta_b y & \text{for } y \in (0, k_0], \\
C_1 y^{\gamma_0} & \text{for } y \in (k_0, \infty).
\end{cases}
\]

In addition all variational inequalities hold.
We provide a verification theorem for both Cases 1 and 2. In Case 1, assume (A1), (A2), and (A3). In Case 2, assume (A1). Then,

\[ v(x_1, x_2, \alpha) = x_1 w_{\alpha}(x_2/x_1) = V(x_1, x_2, \alpha), \alpha = 1, 2. \]

Let \( D = \{(x_1, x_2, 1) : x_2 > k_1 x_1\} \cup \{(x_1, x_2, 2) : x_2 > k_2 x_1\} \). Let

\( \tau^* = \inf\{t : (X^1_t, X^2_t, \alpha_t) \notin D\} \). Then \( \tau^* \) is optimal.
Example: Case 1 ($k_1 < k_2$)

In this example, we take

\[
\begin{align*}
\mu_1(1) &= 0.20, \quad \mu_2(1) = 0.25, \quad \mu_1(2) = -0.30, \quad \mu_2(2) = -0.35, \\
\sigma_{11}(1) &= 0.30, \quad \sigma_{12}(1) = 0.10, \quad \sigma_{21}(1) = 0.10, \quad \sigma_{22}(1) = 0.35, \\
\sigma_{11}(2) &= 0.40, \quad \sigma_{12}(2) = 0.20, \quad \sigma_{21}(2) = 0.20, \quad \sigma_{22}(2) = 0.45, \\
\lambda_1 &= 6.0, \quad \lambda_2 = 10.0, \quad K = 0.001, \quad \rho = 0.50.
\end{align*}
\]

Then, the unique zero of $f(r)$ ($r > 1$) is given by $r_0 = 1.020254$. Using this $r_0$, we obtain $k_1 = 0.723270$ and $k_2 = 0.737920$. Then, we calculate and get $C_1 = 0.11442$, $C_2 = -0.00001$, $C_3 = 0.29121$, $C_4 = 0.00029$, $\eta_3 = 0.985919$, and $\eta_4 = -1.541271$. With these numbers, we verify all variational inequalities required in (A3).
Value Functions $V(x_1, x_2, 1)$ and $V(x_1, x_2, 2)$

Figure 1: Value Functions $V(x_1, x_2, 1)$ and $V(x_1, x_2, 2)$
Example \( (k_1 = k_2) \)

In this example, we take

\[
\begin{align*}
\mu_1(1) &= \mu_1(2) = 0.20, \quad &\mu_2(1) &= \mu_2(2) = 0.25, \\
\sigma_{11}(1) &= \sigma_{11}(2) = 0.30, \quad &\sigma_{12}(1) &= \sigma_{12}(2) = 0.10, \\
\sigma_{21}(1) &= \sigma_{21}(2) = 0.10, \quad &\sigma_{22}(1) &= \sigma_{22}(2) = 0.35, \\
\lambda_1 &= 6.0, \quad &\lambda_2 &= 10.0, \quad &K &= 0.001, \quad &\rho &= 0.50.
\end{align*}
\]

We have \( k_0 = 0.705098 \) and \( C_1 = 0.126431 \). This gives the corresponding value function.
Example ($k_1 = k_2$)

In this example, we take

\begin{align*}
\mu_1(1) &= \mu_1(2) = 0.20, & \mu_2(1) &= \mu_2(2) = 0.25, \\
\sigma_{11}(1) &= \sigma_{11}(2) = 0.30, & \sigma_{12}(1) &= \sigma_{12}(2) = 0.10, \\
\sigma_{21}(1) &= \sigma_{21}(2) = 0.10, & \sigma_{22}(1) &= \sigma_{22}(2) = 0.35, \\
\lambda_1 &= 6.0, & \lambda_2 &= 10.0, & K &= 0.001, & \rho &= 0.50.
\end{align*}

We have $k_0 = 0.705098$ and $C_1 = 0.126431$. This gives the corresponding value function.

A sufficient and necessary condition for $k_1 = k_2$ is $\gamma_0\big|_{\alpha=1} = \gamma_0\big|_{\alpha=2}$. 
Value Function $V(x_1, x_2) = V(x_1, x_2, 1) = V(x_1, x_2, 2)$

Figure 2: Value Function $V(x_1, x_2) = V(x_1, x_2, 1) = V(x_1, x_2, 2)$
Example \((k_1 > k_2)\).

Finally, we take a different set of parameters:

\[
\begin{align*}
\mu_1(1) &= -0.10, \quad \mu_2(1) = 0.20, \quad \mu_1(2) = 0.25, \quad \mu_2(2) = -0.15, \\
\sigma_{11}(1) &= 0.35, \quad \sigma_{12}(1) = 0.15, \quad \sigma_{21}(1) = 0.15, \quad \sigma_{22}(1) = 0.30, \\
\sigma_{11}(2) &= 0.20, \quad \sigma_{12}(2) = 0.10, \quad \sigma_{21}(2) = 0.10, \quad \sigma_{22}(2) = 0.15, \\
\lambda_1 &= 6.0, \quad \lambda_2 = 10.0, \quad K = 0.001, \quad \rho = 0.50.
\end{align*}
\]

In this example, if we apply the same procedure used in Example 1 for \(k_1\) and \(k_2\), we notice some of the variational inequalities in (A3) will be violated. This means the condition \(k_1 < k_2\) does not apply. Based on the symmetry of the problem in \(\alpha = 1\) and \(\alpha = 2\), we switch the set of parameters about \(\alpha = 1\) and \(\alpha = 2\) and obtain \(\tilde{k}_1 = 0.379300\) and \(\tilde{k}_2 = 0.824070\). The ‘new’ value functions \((\tilde{V}(x_1, x_2, 1), \tilde{V}(x_1, x_2, 2))\) can be obtained in a similar way. So are the verification of the variational inequalities in (A3). Then, we switch back to obtain \(k_1 = \tilde{k}_2 = 0.824070\) and \(k_2 = \tilde{k}_1 = 0.379300\). The same for the value functions \((V(x_1, x_2, 1) = \tilde{V}(x_1, x_2, 2)\) and \(V(x_1, x_2, 2) = \tilde{V}(x_1, x_2, 1))\).
Value Functions $V(x_1, x_2, 1)$ and $V(x_1, x_2, 2)$

Figure 3: Value Functions $V(x_1, x_2, 1)$ and $V(x_1, x_2, 2)$
Conclusion

- The main focus of this paper is on a pairs trade selling rule.
- It extends the results of McDonald and Siegel (and Hu and Øksendal) by incorporating models with regime switching.
- It would be interesting to extend the results to include the buying side of optimal timing (in progress).
- It would also be interesting to consider models in which the market mode $\alpha_t$ is not directly observable.

In this case, the Wonham filter can be used for calculation of the conditional probabilities of $\alpha = 1$ given the stock prices up to time $t$. 

Hu and Øksendal. 1998. Optimal time to invest when the price processes are geometric Brownian motions, *Finance and Stochastics*.


