# Stochastic Damping Hamiltonian Systems with State-Dependent Switching

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# Hamiltonian system with damping

- Hamiltonian systems have a wide range of applications and are used as models for classical mechanics, electromagnetic forces, quantum mechanics, etc.
- In physical systems, damping is produced by processes that dissipate the energy stored in the oscillation. Examples include viscous drag in mechanical systems, resistance in electronic oscillators, and absorption and scattering of light in optical oscillators.
- The evolution of the system can be formally described by the Hamilton equation:

$$\ddot{\eta}(t) + c(\eta(t), \dot{\eta}(t))\dot{\eta}(t) + \nabla_x V(\eta(t)) = 0$$

where $c$ is the damping coefficient and $V(x)$ is the potential.

# Examples

- the Duffing oscillator:
  $c(x, y) \equiv c > 0$ and $V(x)$ is a lower bounded polynomial.
- the van der Pol oscillator:
  $c(x, y) = x^2 - 1$, $V(x) = \frac{1}{2}\omega_0^2 x^2$.
- the Liénard oscillator:
  $c(x, y) = f(x)$ and $V(x) = \int_0^x g(u)\mathrm{d}u$ with $f$ and $g$ being appropriate continuously differentiable functions on $\mathbb{R}$.

# Stochastic Hamiltonian Systems with Damping

A stochastic Hamiltonian system with damping can be formally described by

$$\ddot{\eta}(t) + c(\eta(t), \dot{\eta}(t))\dot{\eta}(t) + \nabla_x V(\eta(t)) = \sigma(\eta(t), \dot{\eta}(t))\dot{W}(t),$$

where

- $c$ is the damping coefficient,
- $V$ is the potential,
- $\dot{W}$ denotes the generalized derivative of a standard Wiener process,
- $\sigma\dot{W}$ is the random force.

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2. Eckmann, J.-P. and Hairer, M. (2000). Non-equilibrium statistical mechanics of strongly anharmonic chains of oscillators. *Comm. Math. Phys.*, 212(1):105–164.
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# Stochastic Damping Hamiltonian System with Switching: Motivations

## Weakly coupled oscillators

- Suppose there are infinitely many oscillators indexed by $k \in \mathbb{S} = \{1, 2, \dots\}$.
- At time $t = 0$, only one oscillator, say, $i \in \mathbb{S}$, is active, whose dynamics is given by

$$\ddot{\eta}_i(t) + c_i(\eta_i(t), \dot{\eta}_i(t))\dot{\eta}_i(t) + \nabla_x V_i(\eta_i(t)) = \sigma_i(\eta_i(t), \dot{\eta}_i(t))\dot{W}_i(t),$$

- After a random amount of time, the oscillator $i$ becomes dormant and another oscillator, say, $j \neq i$, becomes active. The dynamics of oscillator $j$ is given by

$$\ddot{\eta}_j(t) + c_j(\eta_j(t), \dot{\eta}_j(t))\dot{\eta}_j(t) + \nabla_x V_j(\eta_j(t)) = \sigma_j(\eta_j(t), \dot{\eta}_j(t))\dot{W}_j(t),$$

  The oscillator $j$ will stay active for another random amount of time until it becomes dormant and another oscillator becomes active.

- ... ...

# Stochastic Damping Hamiltonian System with Switching: Motivations

## Oscillator in random environments

- Physical systems are subject to various random perturbations.
- In particular, the potential function, damping coefficient, and random force may change randomly and abruptly, resulting in *structural changes* for the Hamilton system.

  Examples:
  - nonlinear vibration systems under random excitation,
  - particles or electromagnetic waves propagate through different media.

# Stochastic Damping Hamiltonian System with Switching

- Write $X(t) := \eta(t) \in \mathbb{R}^d$ and $Y(t) := \dot{\eta}(t) \in \mathbb{R}^d$.
- Consider $X(t)$ being the position and $Y(t)$ the momentum of the system.
- $Z = (X, Y)$ satisfies

$$\begin{cases} \mathrm{d}X(t) = Y(t)\mathrm{d}t, \\ \mathrm{d}Y(t) = -\big[c(X(t), Y(t), \Lambda(t))Y(t) + \nabla_x V(X(t), \Lambda(t))\big]\mathrm{d}t \\ \qquad\qquad +\sigma(X(t), Y(t), \Lambda(t))\mathrm{d}B(t), \end{cases} \tag{1}$$

- $B \in \mathbb{R}^d$ is a standard Brownian motion.

- $\Lambda \in \mathbb{S} = \{1, 2, \dots\}$ satisfies

$$\begin{aligned} &\mathbb{P}\{\Lambda(t+\Delta) = l | \Lambda(t) = k, (X(t), Y(t)) = (x, y)\} \\ &= \begin{cases} q_{kl}(x, y)\Delta + o(\Delta), & \text{if } k \neq l, \\ 1 + q_{kk}(x, y)\Delta + o(\Delta), & \text{if } k = l, \end{cases} \end{aligned} \tag{2}$$

  uniformly in $\mathbb{R}^{2d}$, provided $\Delta \downarrow 0$.

- The evolution of the environments depends on the state of the Hamiltonian system.

# The Standing Assumptions

For each $k \in \mathbb{S}$, we assume that

(i) the potential function $V(\cdot, k)$ is bounded below and continuously differentiable on $\mathbb{R}^d$;

(ii) the damping coefficient $c(\cdot, \cdot, k)$ is continuous and for all $N > 0$:

$$\sup\{\|c(x, y, k)\|_{\text{H.S.}} : |x| \leq N, y \in \mathbb{R}^d\} < \infty,$$

and there exist $c$, $L > 0$ such that

$$c^s(x, y, k) \geq cI > 0 \text{ for all } |x| > L \text{ and } y \in \mathbb{R}^d;$$

Here $c^s(x, y, k) := \frac{1}{2}(c(x, y, k) + c^T(x, y, k))$ and $\|\cdot\|_{\text{H.S.}}$ is the Hilbert-Schmidt norm of matrices.

(iii) the random perturbation $\sigma(\cdot,\cdot,k)$ is symmetric, infinitely differentiable and for some $\hat{\sigma} > 0$: $0 < \sigma(x,y,k) \leq \hat{\sigma} I$ over $\mathbb{R}^{2d}$, where $I$ is the $d$-dimensional identity matrix;

(iv) the formal generator of the switching process $Q(x,y) := \big(q_{kl}(x,y)\big)$ is a matrix-valued measurable function on $\mathbb{R}^{2d}$ such that for all $(x,y) \in \mathbb{R}^{2d}$ and $k \in \mathbb{S}$,

- $q_{kl}(x,y) \geq 0$ for $k \neq l$ and $q_{kk}(x,y) = -\sum_{l \in \mathbb{S}\setminus\{k\}} q_{kl}(x,y) \leq 0$;
- there exists a constant $H > 0$ such that

$$\sup_{k \in \mathbb{S}} \sum_{l \in \mathbb{S}\setminus\{k\}} \sup_{(x,y) \in \mathbb{R}^{2d}} q_{kl}(x,y) \leq H.$$

# Some Remarks

- The diffusion coefficient of (1) is **degenerate**.
- The coefficients $\nabla V(x, k)$ and $c(x, y, k)$ are **only continuous**.
- Hence, the hypoellipticity need not hold for (1).
- Besides, $\nabla_x V(x, k)$ and $c(x, y, k)$ perhaps satisfy neither the linear growth nor the Lipschitz conditions.
- The component $\Lambda$ has a countable state space and its switching rates depend on the state $(X, Y)$.

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The existence and uniqueness of a solution and properties such as the strong Feller property of the corresponding Markov process are not obvious.

# In this work

- Weak solution via the martingale problem approach.
- Strong Feller property by the killing technique and a transition probability identity.
- Exponential ergodicity under a Foster-Lyapunov drift condition.
- Large deviation principle.

# The Operators

For all functions $f \in C_c^\infty(\mathbb{R}^{2d} \times \mathbb{S})$, we define the following operator:

$$\mathcal{A}f(x, y, k) := \mathcal{L}_k f(x, y, k) + Q(x, y) f(x, y, k). \tag{3}$$

Here, for each $k \in \mathbb{S}$,

- $\mathcal{L}_k$ is the differential operator:

$$\begin{aligned}\mathcal{L}_k f(x, y, k) := \frac{1}{2}\mathrm{tr}\big(a(x, y, k)\nabla_y^2 f(x, y, k)\big) + \langle y, \nabla_x f(x, y, k)\rangle \\ - \langle c(x, y, k)y + \nabla_x V(x, k), \nabla_y f(x, y, k)\rangle,\end{aligned} \tag{4}$$

  where $a(x, y, k) = \sigma(x, y, k)\sigma(x, y, k)^T$,

- the switching operator $Q(x, y)$ is given by:

$$Q(x, y) f(x, y, k) := \sum_{l \in \mathbb{S}} q_{kl}(x, y)\big(f(x, y, l) - f(x, y, k)\big). \tag{5}$$

# The Probability Space

- Define a metric $\lambda(\cdot,\cdot)$ on $\mathbb{R}^{2d} \times \mathbb{S}$ as

$$\lambda\big((x,y,m),(\widetilde{x},\widetilde{y},\widetilde{m})\big) = |(x,y) - (\widetilde{x},\widetilde{y})| + d(m,\widetilde{m}),$$

  where $d(\cdot,\cdot)$ is the discrete metric.

- Let $\Omega := C([0,\infty),\mathbb{R}^{2d}) \times D([0,\infty),\mathbb{S})$ be endowed with the product topology of the sup norm topology on $C([0,\infty),\mathbb{R}^{2d})$ and the Skorohod topology on $D([0,\infty),\mathbb{S})$

- $(X, Y, \Lambda) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}$ is the coordinate process on $\Omega$.

- Let $\mathcal{F}_t$ be the $\sigma$-field generated by the cylindrical sets on $\Omega$ and set $\mathcal{F} = \bigvee_{t=0}^{\infty} \mathcal{F}_t$.

# The Martingale Problem

For a given $(x, y, k) \in \mathbb{R}^{2d} \times \mathbb{S}$, we say a probability measure $\mathbb{P}^{(x,y,k)}$ on $\Omega$ is a solution to the martingale problem for the operator $\mathcal{A}$ starting from $(x, y, k)$, if

- $\mathbb{P}^{(x,y,k)}((X(0), Y(0), \Lambda(0)) = (x, y, k)) = 1$, and
- for each function $f \in C_c^\infty(\mathbb{R}^{2d} \times \mathbb{S})$,

$$M_t^{(f)} := f(X(t), Y(t), \Lambda(t)) - f(X(0), Y(0), \Lambda(0)) - \int_0^t \mathcal{A}f(X(s), Y(s), \Lambda(s))\mathrm{d}s, \quad t \geq 0$$

is an $\{\mathcal{F}_t\}$-martingale with respect to $\mathbb{P}^{(x,y,k)}$.

# The Auxiliary Process

For each $k \in \mathbb{S}$, let $Z^{(k)}(t) := (X^{(k)}(t), Y^{(k)}(t))$ satisfy the following stochastic differential equation

$$\begin{cases} \mathrm{d}X^{(k)}(t) = Y^{(k)}(t)\mathrm{d}t, \\ \mathrm{d}Y^{(k)}(t) = -\big[c(X^{(k)}(t), Y^{(k)}(t), k)\, Y(t) + \nabla_x V(X^{(k)}(t), k)\big]\mathrm{d}t \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \sigma(X^{(k)}(t), Y^{(k)}(t), k)\mathrm{d}B(t). \end{cases} \tag{6}$$

# Preparatory Results (Wu 2001, SPA)

**Lemma 1.**
*For each $k \in \mathbb{S}$ and for each initial state $z = (x, y) \in \mathbb{R}^{2d}$, (6) admits a unique weak solution $\mathbb{P}_k^{(z)}$, a probability measure on the space $C([0, \infty), \mathbb{R}^{2d})$, and this solution is non-explosive.*

**Lemma 2.**
*For each $k \in \mathbb{S}$, let $\big(P_k(t, z, \cdot)\big)$ be the transition probability family of Markov process $\big((Z^{(k)}(t))_{t \geq 0}, (\mathbb{P}_k^{(z)})_{z \in \mathbb{R}^{2d}}\big)$. For each $k \in \mathbb{S}$, $t > 0$ and $z \in \mathbb{R}^{2d}$, $P_k(t, z, dz') = p_k(t, z, z')dz'$, $p_k(t, z, z') > 0$, $dz'$-a.e. and*

$$z \to p_k(t, z, \cdot) \text{ is continuous from } \mathbb{R}^{2d} \text{ to } L^1(\mathbb{R}^{2d}, dz').$$

*In particular, the process $Z^{(k)}$ is strong Feller.*

# Weak Solution of Each Subsystem

For each $k \in \mathbb{S}$ and $z = (x, y) \in \mathbb{R}^{2d}$, it follows from Lemma 1 that

- $\mathbb{P}_k^{(z)}(Z(0) = z) = 1$
- $\forall f \in C_c^\infty(\mathbb{R}^{2d})$,

$$M_t^{(k)(f)} := f(Z(t)) - f(Z(0)) - \int_0^t \mathcal{L}_k f(Z(s))\mathrm{d}s, \quad t \geq 0$$

is a $\{\mathcal{G}_t\}$-martingale with respect to $\mathbb{P}_k^{(z)}$, where $\mathcal{G}_t$ is the $\sigma$-field generated by the cylindrical sets on $C([0, \infty), \mathbb{R}^{2d})$ up to time $t$.

# Auxiliary Operator

Consider a special $Q$-matrix $\widehat{Q} = (\widehat{q}_{kl})$ given by

$$\widehat{q}_{kl} := \sup_{z \in \mathbb{R}^{2d}} q_{kl}(z) \text{ for } k \neq l, \text{ and } \widehat{q}_{kk} := -\sum_{l \neq k} \widehat{q}_{kl} \text{ for } k \in \mathbb{S}.$$

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Then we can define

$$\widehat{Q}f(k) = \sum_{l \in \mathbb{S}} \widehat{q}_{kl}\big(f(l) - f(k)\big), \quad f \in \mathcal{B}_b(\mathbb{S}).$$

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Then we can define

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Next we introduce an operator $\widehat{\mathcal{A}}$ on $C_c^2(\mathbb{R}^{2d} \times \mathbb{S})$ as follows:

$$\widehat{\mathcal{A}}f(x, y, k) := \mathcal{L}_k f(x, y, k) + \widehat{Q}f(x, y, k), \quad f \in C_c^2(\mathbb{R}^{2d} \times \mathbb{S}).$$

- Write $\omega = (\omega_1, \omega_2) \in \Omega := \Omega_1 \times \Omega_2$ with $\Omega_1 := C([0,\infty), \mathbb{R}^{2d})$ and $\Omega_2 := D([0,\infty), \mathbb{S})$.
- For each $k \in \mathbb{S}$, there exists a unique martingale solution $\mathbb{Q}^{(k)} \in \mathscr{P}(\Omega_2)$ for the operator $\widehat{Q}$ starting from $k$.
- Define $\tau_0(\omega_2) \equiv 0$, and for $n \geq 1$,

$$\tau_n(\omega_2) := \inf\{t > \tau_{n-1}(\omega_2) : \Lambda(t, \omega_2) \neq \Lambda(\tau_{n-1}(\omega_2), \omega_2))\}.$$

- $\mathbb{Q}^{(k)}\{\lim_{n\to\infty} \tau_n = +\infty\} = 1$.
- For each $n \geq 1$, there exists a $\mathbb{P}^{(Z(\tau_n))}_{\Lambda(\tau_n)} \in \mathscr{P}(\Omega_1)$ s.t. for any $f \in C^2_c(\mathbb{R}^{2d})$,

$$f(Z(t)) - f(Z(\tau_n)) - \int_{\tau_n}^{t} \mathcal{L}_{\Lambda(\tau_n)} f(Z(s)) \mathrm{d}s, \quad t \geq \tau_n$$

is a martingale under $\mathbb{P}^{(Z(\tau_n))}_{\Lambda(\tau_n)}$.

# Weak Solution: Special Case

**Theorem 3.**

*For any given* $(x, y, k) \in \mathbb{R}^{2d} \times \mathbb{S}$, *there exists a unique martingale solution* $\widehat{\mathbb{P}}^{(x,y,k)}$ *on* $C([0,\infty), \mathbb{R}^{2d}) \times D([0,\infty), \mathbb{S})$ *for the operator* $\widehat{\mathcal{A}}$ *starting from* $(x, y, k)$.

# Idea of Proof

## Stroock–Varadhan Piecing-together Method

- For any given $(z, k) := (x, y, k) \in \mathbb{R}^{2d} \times \mathbb{S}$, we define a sequence of probability measures on $(\Omega, \mathcal{F})$ as follows:

$$\mathbb{P}^{(1)} = \mathbb{P}_k^{(z)} \times \mathbb{Q}^{(k)},$$

and for $n \geq 1$,

$$\mathbb{P}^{(n+1)} = \mathbb{P}^{(n)} \otimes_{\tau_n} \left( \mathbb{P}_{\Lambda(\tau_n)}^{(Z(\tau_n))} \times \mathbb{Q}^{(\Lambda(\tau_n))} \right),$$

where $\tau_n(\omega) = \tau_n(\omega_1, \omega_2) := \tau_n(\omega_2)$.

# Toward the General Case

- For $t > 0$, $k \in \mathbb{S}$, and $A \subset \mathbb{S}$, let

$$n(t, A) := \sum_{s \le t} \mathbf{1}_{\{\Lambda(s) \in A, \Lambda(s) \neq \Lambda(s-)\}}.$$

  This is a random counting measure on $[0, \infty) \times \mathbb{S}$.

- Also, for $k \in \mathbb{S}$ and $A \subset \mathbb{S}$, we define

$$\nu(k; A) := \sum_{l \in A \setminus \{k\}} \widehat{q}_{kl}.$$

- Then we can show that

$$\widetilde{n}(t, A) := n(t, A) - \int_0^t \nu(\Lambda(s-); A) \mathrm{d}s$$

  is a martingale measure with respect to $\widehat{\mathbb{P}}^{(x,y,k)}$.

- Define

$$g(k,l,z) := \begin{cases} \frac{q_{kl}(z)}{\widehat{q}_{kl}} \mathbf{1}_{\{\widehat{q}_{kl}>0\}}, & \text{if } k \neq l, z \in \mathbb{R}^{2d}, \\ 0, & \text{if } k = l, z \in \mathbb{R}^{2d}, \end{cases}$$

and

$$\xi(t) := \int_{[0,t]\times\mathbb{S}} [g(\Lambda(s-), l, Z(s)) - 1]\widetilde{n}(\mathrm{d}s, \mathrm{d}l), \quad t \geq 0$$

- Observe that $\xi$ is a martingale under $\widehat{\mathbb{P}}^{(x,y,k)}$.

- Define

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- Observe that $\xi$ is a martingale under $\widehat{\mathbb{P}}^{(x,y,k)}$.

**Lemma 4.**

*The process $M_\cdot$ defined by*

$$M_t := 1 + \int_0^t M_{s-} \mathrm{d}\xi(s), \quad t \geq 0,$$

*is a square-integrable martingale with $\widehat{\mathbb{E}}[M_t] = 1$ for all $t \geq 0$.*

# Martingale Solution: the General Case

**Theorem 5.**

*For any given* $(z, k) \in \mathbb{R}^{2d} \times \mathbb{S}$, *there exists a unique martingale solution* $\mathbb{P}^{(z,k)}$ *on* $\Omega$ *for the operator* $\mathcal{A}$ *starting from* $(z, k)$. *Consequently for any initial data* $(z, k)$, *the system* (1)–(2) *has a unique weak solution.*

# Martingale Solution: the General Case

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- First for each $t \geq 0$ and each $A \in \mathcal{F}_t$, define

$$\mathbb{P}_t^{(z,k)}(A) = \int_A M_t \, \mathrm{d}\widehat{\mathbb{P}}^{(z,k)}.$$

- $\{\mathbb{P}_t^{(z,k)}\}_{t\geq 0}$ is a consistent family of probability measures by Lemma 4.
- By Tulcea's extension theorem, there exists a unique probability measure $\mathbb{P}^{(z,k)}$ on $(\Omega, \mathcal{F})$ s.t. $\mathbb{P}^{(z,k)} = \mathbb{P}_t^{(z,k)}$ on $\mathcal{F}_t$, $\forall t \geq 0$.
- This $\mathbb{P}^{(z,k)}$ is the desired martingale solution starting from $(z,k)$.

# Strong Feller Property

- For $f \in \mathcal{B}_b(\mathbb{R}^{2d} \times \mathbb{S})$, set

$$P_t f(z,k) := \mathbb{E}_{z,k}[f(X(t), Y(t), \Lambda(t))], \quad t \geq 0, (z,k) \in \mathbb{R}^{2d} \times \mathbb{S}.$$

- The semigroup $\{P_t\}_{t \geq 0}$ is *strong Feller* if it maps $\mathcal{B}_b(\mathbb{R}^{2d} \times \mathbb{S})$ into $C_b(\mathbb{R}^{2d} \times \mathbb{S})$ for each $t > 0$.
- Usual approaches: coupling method, PDE, etc.

# Related Work

- Z. and Yin (2009), On strong Feller, recurrence, and weak stabilization of regime-switching diffusions, *SICON*, **48**(3), 2003–2031.
- Xi and Yin (2013), The strong Feller property of switching jump-diffusion processes, *SPL*, **83**, 761–767.
- Shao (2015), Strong solutions and strong Feller properties for regime-switching diffusion processes in an infinite state space. *SICON*, **53** (2015), 2462–2479.
- Xi and Z. (2017), On Feller and Strong Feller Properties and Exponential Ergodicity of Regime-Switching Jump Diffusion Processes with Countable Regimes, *SICON*, **55**, 1789–1818.
- Kunwai and Z. (2020), On Feller and strong Feller properties and irreducibility of regime-switching jump diffusion processes with countable regimes, *NAHS*, **38**, 100946.
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Approach: coupling method, PDE
Conditions: uniform ellipticity, Lipschitz or Hölder coefficients.

# A Useful Result

## Proposition

*The semigroup* $\{P_t\}_{t\geq 0}$ *is strong Feller if and only if for any* $A \times \{l\} \in \mathcal{B}(\mathbb{R}^{2d} \times \mathbb{S})$, *the function* $(z,k) \mapsto P(t,(z,k),A \times \{l\})$ *is lower semicontinuous.*

S. Meyn and R. L. Tweedie. (2009), *Markov chains and stochastic stability*. Cambridge University Press, Cambridge, second edition.

# Killing

- For each $k \in \mathbb{S}$, the operator $\mathcal{L}_k$ of (4) uniquely determines a process $Z^{(k)}$.
- The killed process $\widetilde{Z}^{(k)}$:

$$\mathbb{E}_k[f(\widetilde{Z}^{(k)(z)}(t))]$$
$$= \mathbb{E}_k\left[f(Z^{(k)(z)}(t)) \exp\left\{\int_0^t q_{kk}(Z^{(k)(z)}(s))\mathrm{d}s\right\}\right].$$

- The killed process $\widetilde{Z}^{(k)}$ is strong Feller.

- Denote by $\{\widetilde{P}^{(k)}(t,z,A) : t \geq 0, z \in \mathbb{R}^{2d}, A \in \mathcal{B}(\mathbb{R}^{2d})\}$ the sub-transition probability families of the killed process $\widetilde{Z}^{(k)}$.
- We have

$$
\begin{aligned}
&P(t,(z,k),A\times\{l\})\\
&\quad=\delta_{kl}\widetilde{P}^{(k)}(t,z,A)+\sum_{m=1}^{+\infty}\int_{0<t_1<\cdots<t_m<t}\cdots\int\sum_{\substack{l_1\in\mathbb{S}\setminus\{l_0\},l_2\in\mathbb{S}\setminus\{l_1\},\cdots,l_m\in\mathbb{S}\setminus\{l_{m-1}\},\\ l_0=k,\ l_m=l}}\\
&\qquad\int_{\mathbb{R}^{2d}}\cdots\int_{\mathbb{R}^{2d}}\widetilde{P}^{(l_0)}(t_1,z,\mathrm{d}z_1)q_{l_0l_1}(z_1)\widetilde{P}^{(l_1)}(t_2-t_1,z_1,\mathrm{d}z_2)\cdots\\
&\qquad\times q_{l_{m-1}l_m}(z_m)\widetilde{P}^{(l_m)}(t-t_m,z_m,A)\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_m.
\end{aligned}
$$

- $\widetilde{P}^{(k)}(t,z,A)$ and every term in the series are lower semicontinuous with respect to $z$ whenever $A \in \mathcal{B}(\mathbb{R}^{2d})$.

# Strong Feller Property

**Theorem 6.**
*The process* $(X, Y, \Lambda)$ *has the strong Feller property.*

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## Remarks

Compared with the aforementioned references, in this work

- the diffusion matrix is degenerate,
- no Lipschitz or Hölder continuity on the coefficients $c$ and $\nabla V$,
- the switching rates $q_{kl}$ are only assumed to be bounded and measurable,
- different approach.

# Exponential Ergodicity

- The Markov process $(Z, \Lambda)$ is said to be *exponentially ergodic* if there exist a probability measure $\pi(\cdot)$, a constant $\theta < 1$ and a finite-valued function $\Theta(x, k)$ such that

$$\|P(t, (z, k), \cdot) - \pi(\cdot)\|_{TV} \leq \Theta(z, k)\theta^t$$

for all $t \geq 0$ and all $(z, k) \in \mathbb{R}^{2d} \times \mathbb{S}$.

**Theorem 7.**

*In addition to the standing assumptions, suppose that*

(a) *the matrix $Q$ is* irreducible *on $\mathbb{R}^{2d}$ in the following sense: for any distinct $k, l \in \mathbb{S}$, there exist $r \in \mathbb{N}$, $k_0, k_1, \ldots, k_r \in \mathbb{S}$ with $k_i \neq k_{i+1}$, $k_0 = k$ and $k_r = l$ such that the set $\{z \in \mathbb{R}^{2d} : q_{k_i k_{i+1}}(z) > 0\}$ has positive Lebesgue measure for $i = 0, 1, \ldots, r-1$.*

(b) *there exists a nonnegative function $\widetilde{V} \in C^2(\mathbb{R}^{2d} \times \mathbb{S}; \mathbb{R}_+)$ satisfying $\widetilde{V}(z,k) \to \infty$ as $|z| \vee k \to \infty$ as well as a Foster-Lyapunov drift condition:*

$$\mathcal{A}\widetilde{V}(z,k) \leq -\alpha \widetilde{V}(z,k) + \beta, \quad (z,k) \in \mathbb{R}^{2d} \times \mathbb{S},$$

*where $\alpha$, $\beta > 0$ are constants.*

*Then Markov process $(Z(\cdot), \Lambda(\cdot))$ is exponentially ergodic.*

## Remark

Sufficient conditions on the potential function $V$, the damping coefficient $c$, and the switching rate matrix $Q(\cdot)$ can be derived.

# An Example

- $d = 1$ and $\mathbb{S} = \{1, 2, \dots\}$.
- $c(x, y, k) = 0$ for $|x| \leq 1$ and $c(x, y, k) = \frac{\sqrt{k}}{2}$ for $|x| \geq 2$ and $(y, k) \in \mathbb{R} \times \mathbb{S}$.
- $V(\cdot, k) \in C^1(\mathbb{R})$ and satisfies $V(x, k) = 0$ for $|x| \leq 1$ and $V(x, k) = (2 - \frac{1}{k})x^4$ for $|x| \geq 2$ and $k \in \mathbb{S}$ (so the potential kicks in only when the particle is outside the ball $\{|x| < 1\}$).
- for $(x, y) \in \mathbb{R}^2$

$$q_{kj}(x, y) := \frac{k}{2^j(k + (1 + x^2 + y^2)^{-1})} \text{ for } j \neq k,$$

$$q_{kk}(x, y) := -\sum_{j \neq k} q_{kj}(x, y).$$

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- for $(x, y) \in \mathbb{R}^2$

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$\rightsquigarrow$ The system is exponentially ergodic.

- occupation empirical measure

$$L_t(\cdot) := \frac{1}{t}\int_0^t \delta_{(Z(s),\Lambda(s))}(\cdot)\mathrm{d}s,$$

  where $\delta_\cdot$ denotes the Dirac measure.

- the process-level empirical measures

$$R_t(\cdot) := \frac{1}{t}\int_0^t \delta_{(Z(s+\cdot),\Lambda(s+\cdot))}(\cdot)\mathrm{d}s,$$

  where $(Z(s+\cdot),\Lambda(s+\cdot))$ denotes the path $[0,\infty) \ni t \to (Z(s+t),\Lambda(s+t)) \in \Omega = C([0,\infty),\mathbb{R}^{2d}) \times D([0,\infty),\mathbb{S})$.

- For each $t > 0$, $L_t : \Omega \mapsto \mathscr{P}(\mathbb{R}^{2d} \times \mathbb{S})$ and $R_t : \Omega \mapsto \mathscr{P}(\Omega)$.

- Question: Do they satisfy LDPs?

## Theorem 8 (Wu, 2001).

*Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Z_t, \{\mathbb{P}_z, z \in E\})$ be a Markov process valued in a Polish space $E$ with semigroup $\{P_t\}$. Suppose $\{P_t\}$ is strong Feller and irreducible. Then the following are equivalent:*

(a) *$\mathbb{P}_z\{L_t \in \cdot\}$ satisfies the LDP on $\mathscr{P}(E)$ w.r.t. the $\tau$-topology with the rate function $J$ (the Donsker-Varadhan level-2 entropy functional); uniformly for $z$ in compacts.*

(b) *$\mathbb{P}_z\{R_t \in \cdot\}$ satisfies the LDP on $\mathscr{P}(\Omega)$ w.r.t. the $\tau_p$-topology with the rate function $H$ (the Donsker-Varadhan level-3 entropy functional); uniformly for $z$ in compacts.*

(c) *the process satisfies the* hyper-exponential recurrence *property: for any $\lambda > 0$, there exists some compact $\mathbf{K} \subset\subset E$ such that for any $\mathbf{K}' \subset\subset E$,*

$$\sup_{z \in \mathbf{K}'} \mathbb{E}_z[\exp(\lambda \tau_{\mathbf{K}}(T))] < \infty,$$

*where $\tau_{\mathbf{K}}(T) := \inf\{t \geq T : Z_t \in \mathbf{K}\}$.*

## Corollary 9.

*Suppose there exists a norm-like function* $1 \leq W(z,k)$ *satisfying*

$$\lim_{|z|+k\to\infty} \frac{\mathcal{A}W(z,k)}{W(z,k)} = -\infty. \tag{7}$$

*Then the process* $(Z,\Lambda)$ *possesses a unique invariant measure* $\pi \in \mathscr{P}(E)$. *Moreover, for any* $\lambda > 0$, *we can find a compact* $\mathbf{K} \subset\subset \mathbb{R}^d \times \mathbb{S}$ *such that for any* $\mathbf{K}' \subset\subset \mathbb{R}^d \times \mathbb{S}$ *and* $T \geq 0$, *we have*

$$\sup_{(z,k)\in\mathbf{K}'} \mathbb{E}_{z,k}[\exp\{\lambda\tau_{\mathbf{K}}(T)\}] < \infty,$$

*where* $\tau_{\mathbf{K}}(T) := \inf\{t \geq T : (Z(t),\Lambda(t)) \in \mathbf{K}\}$. *Consequently* $L_t, R_t$ *satisfy the large deviation principles of Theorem 8.*

# The van der Pol equation

Let $d = 1$ and $\mathbb{S} = \{1, 2\}$. For $(x, y, k) \in \mathbb{R}^2 \times \{1, 2\}$, define

$$c(x, y, k) = \alpha(k)(x^2 - 1), \quad V(x, k) = \frac{1}{2}\beta(k)x^2,$$

$$Q(x, y) := (q_{kl}(x, y)) = \begin{pmatrix} -\exp(-|x|^3) & \exp(-|x|^3) \\ \frac{\widetilde{H}}{|x|^2+|y|^2+1} & -\frac{\widetilde{H}}{|x|^2+|y|^2+1} \end{pmatrix},$$

where $\alpha(1) = 1$, $\alpha(2) = 2$, $\beta(1) = 2$ and $\beta(2) = 1$, and $\widetilde{H}$ is an arbitrary positive constant. Moreover, let $\sigma(x, y, k)$ be "nice" so that all assumptions are satisfied.

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Detailed computations using appropriate Lyapunov function reveal that the van der Pol system is exponentially ergodic and satisfies the large deviation principles of Theorem 8.

# Overdamped Langevin equation

Consider the overdamped Langevin equation

$$\mathrm{d}X(t) = -\nabla_x V(X(t), \Lambda(t))\mathrm{d}t + \mathrm{d}W(t), \tag{8}$$

in which $W$ is a 1-dimensional standard Brownian motion, the potential is given by

$$V(x,1) = \frac{x^4}{4}, \quad V(x,2) := (x^2+1)\mathbf{1}_{\{|x|\leq 1\}} + 2|x|\mathbf{1}_{\{|x|>1\}},$$

and $\Lambda \in \mathbb{S} = \{1,2\}$ is the switching component with generator $Q(x) = (q_{kl}(x))$:

$$Q(x) = \begin{pmatrix} -1 & 1 \\ |x| & -|x| \end{pmatrix}. \tag{9}$$

- Using appropriate Lyapunov function, we can verify that the system (8)–(9) is exponentially ergodic and hyper-exponentially recurrent (and thus it satisfies the LDPs of Theorem 8).
- However, the subsystem

$$\mathrm{d}X^{(2)}(t) = -\nabla_x V(X^{(2)}(t), 2)\mathrm{d}t + \mathrm{d}W(t) \tag{10}$$

is exponentially ergodic but not hyper-exponentially recurrent.

Thank you very much!